

welcome students to iit palm problem solving session our topic is probability and this is lecture number two in the last class we have given you two formula so let us recollect them one is that if we have n identical balls which are to be kept in k boxes such that no box will remain empty then the number of possible arrangements is $n - 1 \text{ C } k - 1$ for example if we have three identical balls to be kept in two boxes such that no box is empty then there are two possible ways of doing it one in the first box and two in the second box or two in the first box and one in the second box and the second formula was similar setup n identical balls to be kept in k boxes such that some boxes may remain empty then the number of possible arrangements is $n + k - 1 \text{ C } k - 1$ example three balls two boxes therefore possible arrangements are $0 \ 3 \ 1 \ 2 \ 2 \ 1$ and $3 \ 0$ that is equal to 4 and we have $3 + 2 - 1 \text{ C } 2 - 1$ is equal to $4 \text{ C } 1$ is equal to four

so application in how many ways you can choose five numbers a, b, c, d and e such that each one is greater than 0 and $a + b + c + d + e$ is equal to 20 then the number of possible solutions is as you can well understand it is going to be $20 - 1 \text{ C } 5 - 1$ is equal to $19 \text{ C } 4$ however if a, b, c, d and e are greater than equal to 0 then the number of possible solutions is $20 + 5 - 1 \text{ C } 5 - 1$ is equal to $24 \text{ C } 4$ and how to get that we can assume that we have 20 different 20 balls which are identical or we can say i have 1 1 1 up to 20 times and we are splitting them into five compartments by drawing such lines which i have explained in the last class then this sum of ones in a particular box will give you the corresponding number since there are 20 ones the sum will always be 20 and each one of them you can say this is a this is b and like that this is going to be e

so there we can understand that we can solve these problems using the above formula now let us consider a slightly difficult problem

so problem in how many ways you can choose five numbers n_1, n_2, n_3, n_4 and n_5 such that n_i greater than 0 for all i is equal to 1 to 5 and $n_1 < n_2 < n_3 < n_4$ and that is less than n_5 and $\sum n_i$ is equal to one to five is equal to twenty

so this problem is slightly different as you can understand here we are looking at all the five numbers will have to be different that is a number cannot be repeated they are all greater than 0 and their sum is 20.

so let us understand the problem one possible solution is one two three 4 and 10 this is a possible solution because all 5 are distinct but 1 2 4 4 9 is not a solution since 4 is repeated

so i hope the problem is clear to you

so let us go for solving it solution note that smallest possible value for n_1 is equal to 1 for n_2 is equal to 2 because they are not n_2 cannot be one since we know n_1 is smaller than n_2 and similarly for n_5 this is is equal to five therefore let us define five new variables x_1, x_2, x_3, x_4 and x_5 as follows x_1 is equal to $n_1 - 1$ x_2 is equal to $n_2 - 2$ x_3 is equal to $n_3 - 3$ x_4 is equal to $n_4 - 4$ and x_5 is equal to $n_5 - 5$ therefore each x_i is greater than equal to 0 and $x_1 < x_2 < x_3 < x_4 < x_5$ hence the problem becomes to choose five numbers x_1, x_2, x_3, x_4 and x_5 such that all are greater than equal to 0 and $x_1 + x_2 + x_3 + x_4 + x_5$ is equal to $n_1 - 1 + n_2 - 2 + n_3 - 3 + n_4 - 4 + n_5 - 5$ is equal to $\sum n_i$ is equal to 1 to 5 minus 1 plus 2 plus 3 plus 4 plus 5 is equal to 20 minus 15 is equal to 5.

so we can start as follows x_1, x_2, x_3, x_4, x_5 their sum will

have to be five

so one possible solution is zero zero zero zero and five this gives us the solution 1 2 3 4 and 10 next one is 0 0 0 1 4 therefore this gives us the solution 1 2 3 5 and 9.

0 0 0 2 3 therefore we get the solution 1 2 3 6 and 8 since we cannot decrease from x_5 and give it to x_4

so we go like this we put one here therefore x_4 the smallest value is going to be 1 therefore we are left with a 3 here and therefore the solution that we get is 1 2 4 5 and 8 next one is 0 0 1 we decrease from 1 from here and add it here therefore we get 2 2 therefore the solution is 1 2 4 6 and 7.

next what we can do we now make this to be 1 therefore we get 0 1 x_3 cannot be less than 1

so smallest value is one x_4 again we give one and we give two here

so that makes it five and corresponding solution is one three four 5 and 7 and finally we get 1 1 1 1 and 1 and the corresponding solution is 2 3 4 5 and 6 note that all of them will sum up to 20 therefore number of possible solutions is seven i hope you understood the technique but let me solve a very similar problem

so that you understand it

so problem is in how many ways you can choose four numbers in 1 less than n_2 less than n_3 and less than n_4 that is they are all distinct a n_i greater than 0 for all i is equal to 1 2 3 and 4 and $\sum n_i$ is equal to 1 to 4 is equal to 16 that is the problem

so again as before we define x_1 is equal to $n_1 - 1$ x_2 is equal to $n_2 - 2$ x_3 is equal to $n_3 - 3$ and x_4 is equal to $n_4 - 4$ therefore such that each x_i is greater than equal to 0 x_1 less than equal to x_2 less than equal to x_3 less than equal to x_4 and $\sum x_i$ is equal to 16 minus 10 is equal to 6 therefore we go as follows x_1 x_2 x_3 and x_4 again we generate them very systematically we get 0 0 0 6 0 0 1 5 0 0 2 4 0 0 3 3 0 1 1 4 0 1 2 3 1 1 1 3 1 1 2 2 and we can see that we can also do 0 2 2 2

so we got 1 2 3 4 5 6 7 8 9.

so nine possible solutions i leave it with you that given the arrangement of x_1 x_2 x_3 and x_4 you try to find out what is going to be the set n_1 n_2 n_3 and n_4 now one problem is how do you know that this is a complete set it could happen that you might have missed some of them therefore it needs a mathematical way of doing this this you can do using binomial theorem i want you to think about it in some of the later lectures i shall take up the problem and i will show you how you will be confident that you have taken care of all the possible arrangements of x_1 x_2 x_3 and x_4 ok

so further time being let us now focus on probability we know that we talk about probability when the underlying experiment is random therefore the sample space Ω is known and we need to compute probability of what that is the question if we look at some of the random experiments that we described in the last class then we see that suppose we are looking at coin tossing say five times we may look at what is the probability of two heads or say something like what is the probability of the number of tails is prime similarly if you remember we talked about the problem passengers with bags

so some of the probabilities they that we may look for is what is the probability that number of bags is even or say number of passengers is odd etcetera if you analyze these type of problems we understand that given sample space Ω we are looking at a subset of it and we are trying to find out its probability

so this is what is very important and in mathematical terms we call it an event

what is an event and event is a subset of the sample space Ω why consider the coin tossing problem and suppose we toss a coin three times and we want to compute that number of heads is odd we have seen that there are eight possible outcomes of which we are looking at number of heads is odd that is number of head is one that can be done in this following ways $h t t$ $t h t$ and $t t h$ and number of head is three that is in one way $h h h$ therefore we are looking at the probability of the subset $h t t$ $t h t$ $t t h$ and $h h h$ ok

so now you understand the concept of an event some definitions these subsets of cardinality one are called elementary events that is the outcome of one individual trial is called an elementary event therefore if we throw a die then number of elementary events is six that is one two three up to six an event comprising of more than one elementary event is called a compound event example suppose Ω is the set $1 2 3 4 5 6 7 8 9$ how many compound events are possible since cardinality of Ω is equal to 9 therefore there are 2 to the power 9 possible subsets of which one is ϕ that is a null set and nine are elementary events therefore number of compound events is 2 to the power 9 minus 9 plus 1 is equal to 512 minus 10 is equal to 502

so so many compound events are possible some other definitions two events u one and e two are said to be disjoint if $u \cap e = \phi$ or empty set example throwing a die e one is getting a number less than equal to 3 e two is getting a number greater than four you can see that these are disjoint a sequence of events say $u_1 e_2 e_3 \dots e_k$ is said to be mutually exclusive if $e_i \cap e_j = \phi$ for all i, j belonging to $1 2 3$ up to k and $i \neq j$ another important definition is two events a and b are said to be independent if probability $a \cap b$ is equal to probability of a into probability of b now you may ask me what is probability of an event that is the question

so probability is a mapping from the power set of Ω to $[0, 1]$ that is if a is a subset of Ω then $P(a)$ is the probability associated with the event a is a number p such that $0 \leq p \leq 1$ where p satisfies the following $P(a) \geq 0$ for all $a \subset \Omega$ $P(\Omega) = 1$ and see if a_1, a_2, \dots, a_k are mutually exclusive then $P(a_1 \cup a_2 \cup \dots \cup a_k) = \sum_{i=1}^k P(a_i)$

so these are the basic definitions which allow us to compute probability in general given a set a which is a subset of Ω probability of a is computed as number of elements in a divided by the cardinality of Ω example throwing a die and probability of getting an even number is cardinality of $\{2, 4, 6\}$ divided by cardinality of Ω is equal to three by six is equal to half suppose probability of getting a head is p when $0 \leq p \leq 1$ then what is the probability of getting two heads in three tosses how to obtain such a probability in order to solve that let us first understand a few properties

so show that probability of a complement is equal to $1 - P(a)$ this is true since $a \cup a^c = \Omega$ therefore $1 = P(\Omega) = P(a \cup a^c) = P(a) + P(a^c)$ therefore probability of a complement is equal to $1 - P(a)$ therefore when we are tossing a coin if a is equal to getting a head then a^c is equal to getting a tail therefore if probability of head is equal to p then probability of tail is equal to $1 - p$ now consider getting two heads in three tosses this event can be broken down into probability of $h h t$ plus probability of $h t h$ plus probability of $t h h$ now what is the probability of $h h t$ that means probability of getting of first toss to be head the second toss to be head and the third toss to be tail what is the probability probability of getting a head in the first

probability of a into probability of b is equal to 1 minus probability of a into 1 minus probability of b is equal to probability of a complement into probability of b complement therefore a complement and b complement are independent okay friends i stop here today in the next class i shall start with events and will solve several problems involving algebra of events and to see how to get probabilities of different events okay friends thank you you

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