

welcome students to iit problem solving session on probability this is lecture number one

so in this session in six lectures we shall solve many difficult problems involving probability since this is a problem solving session we shall not focus much on the theory i assume that you are already familiar with the theory and if you are not then i will suggest you to go to iit pal series of lectures and listen to the lectures on probability or you go back to your test books to revise the basic concepts of probability however in this lectures i shall touch upon the most fundamental aspects of probability

so that you can understand what is going on

so let us start when we talk about probability we look at random experiments

so what is a random experiment a random experiment can be characterized as follows it can be carried out any number of times under the identical condition if these experiments the possible set of outcomes is known but the outcome of any individual trial is unknown till it is executed for example tossing a coin we know that the possible set of outcomes is t and h that is tail and head but until the outcome of a toss comes out we do not know whether it is going to be a head or a tail similarly throwing a die the possible set of outcomes is one two three four five six because if you throw a die the outcome is going to be one of these six numbers but we do not know which one will come until we throw the die and see the result tossing a coin three times and noting the sequence of outcomes

so the set of possible outcomes is all three are heads head head then a tail head tail head head tail tail tail head head tail head tail tail tail and then ahead and tail tail tail as you know there can be eight different sequences because there are two possible outcomes for each trial and therefore we get two cube that is eight to be the set of possible outcomes notation this set of possible outcomes is denoted as Ω or S and this is called the sample space ok

so with that background let us solve one problem

so the random experiment that we are describing which is as follows pick a number from the set one two three four five six at random without replacement without replacement means if you take out a number from this set you do not put it back into the set now if the number picked is a prime number then stop else you pick a second number that means again you are picking from the set of which one element has already gone therefore from the rest of the prime numbers you pick a second number if it is a prime you stop or if the two numbers picked are both even then stop else pick a third number that means now there are only four elements in the set because one has already been picked in the first attempt one has already been picked in the second attempt now out of the remaining four you pick a third number

so this is the random experiment the problem is construct the sample space how to solve this problem

so let us attempt that since initially there are six numbers in the first attempt we can choose any one of them

so if i start from here let me write first attempt and what are the possible outcomes one two three four five and six what our experiment says it says that if it is a prime then we stop there therefore we stop at this point if we get a 2 if we get a 3 or if we get a 5 in these 3 cases our output is only one digit namely two or three or five now if i get a one or four or six we go for the second peak

so if we have got one in the second attempt what we can get we can get any of the remaining five and therefore we can draw the graph like this it can be a two three four five six if we have got a four then in the second attempt we can get

one 2 3 5 or 6 and if we get a 6 then in the third attempt we can get 1 2 3 4 and 5 now let us go back to the experiment it says that if this is a prime then we stop or if the two numbers that we have picked up both are even then we stop else we continue for the third attempt

so let us look at when we stop

so this is where we stop this is where we stop and this is where we stop therefore in these cases the outcomes are 1 2 1 3 one five

so let me circle them similarly from here we stop at 2 we stop at 3 we stop at 5 because these are prime numbers and also we stop at six because the two numbers that have been picked are even therefore what are the outcomes

so far we get 4 2 4 3 4 5 4 6 and let me circle them and similarly for the third case we stop if we get 6 2 6 3 6 4 and 6 5 as per the rule of the experiment we stop

so in the remaining cases we now proceed to pick the third number and suppose we got these 1 4 then from 1 4 we can get 2 3 5 and 6 from 1 6 we get 2 3 4 and 5 from 4 1 we get 2 3 five and six and from six one we get two three four and five

so how many sample points are there one length we have two three five length of 2 we have 1 2 3 4 5 6 7 8 9 10 11 and in this case we have 4 plus 4 plus 4 plus 4 that means 16

so total 30 i am not doing it but you can easily understand the point here is going to be 1 4 2 1 4 3 etcetera and like that you can compute the inter set of 30 possible outcomes that we can get for this random experiment let us solve another problem the random experiment is as follows suppose in a bus stop there are four passengers each passenger may have one two or no bags that means this is zero you come to the bus stop and pick up one or two or three or four passengers and each passenger takes his or her bags describe the sample space as a pair of numbers number of passengers and number of bags

so your sample space are made of points of this nature there are two components one is the number of passengers you picked and the number of bags you picked

so question is describe the sample space how to proceed

so in a very similar way you start from here number of passengers may be 1 2 3 or 4 if the number of passenger is 1 the number of bags that you might pick is 0 1 or 2 therefore the points belonging to the omega is equal to 1 0 1 1 1 two now suppose you pick up two passengers how many bags then you may have to pick it may be zero it may be one when one of them has one bag it may be two one may have zero bag other may have two bags or both of them have one bag each three or four therefore the corresponding elements in the sample space are two zero two one two two two three and two four in a similar way if we go from three we will get 3 0 3 1 up to 3 6 therefore the points are going to be 3 0 3 1 three two three three three four three five and three 6 and similarly from 4 you may get 4 0 4 1 4 2 4 3 4 4 5 6 7 and eight

so total number of points in the sample space that is the cardinality of omega is equal to 3 plus 5 plus 7 plus 9 is equal to 20 4.

now the question is why do you need to compute the sample space this is important because we will later see that we will try to associate probabilities with some subset of the possible outputs and for that we need what is the size of the subset that we are interested in and what is the total size of the sample space as we will see later therefore construction of the sample space is very important now

so far we have seen that the sample spaces are finite but it is not necessary that this sample space are going to be always finite for example let us consider the following example you toss a coin repeatedly until you get a head count the number of tosses required to get the first head describe the sample space

so number of tosses if it is one that means in the first toss you got a head if it is two that means in the first toss you get a tail and that is followed by head if it is 3 then you get 2 tails followed by a head if it is in that means you got first n minus one trials all of them are tails followed by a head and n can be any positive integer therefore your Ω is going to be 1 2 3 up to infinity that is the set of all possible integers which we know that is infinite although it is infinite it is countable there may be experiment with uncountable sample space

so example suppose we pick a point from the interval zero to one that means we want to pick up a real number in the interval in the open interval zero to one how many possible ways of choosing it the number of possible ways of selection is infinite but more importantly it is not countable i hope you understand the difference between countable or uncountable

so to make it easy an infinite set Ω is said to be countable if there is a one to one mapping the elements of Ω and this set of natural numbers

so in the earlier example since the set itself is one two three up to infinity this natural mapping is existing there anyways if such a mapping does not exist then it is uncountable why therefore the interval zero to one is uncountable because between any two real numbers say θ .

2 and θ .

35 again we can have infinitely many possible real numbers therefore we cannot make a one-to-one mapping between the set of positive integers and the Ω why i mentioned this because towards the end of this series of lectures i shall do some examples where the Ω is uncountable and that is for your knowledge that in such cases how do you compute the probability ok

so let us consider the following random experiments suppose we have 10 identical red balls we need to put them in three boxes such that none of the boxes is empty how many possible ways of putting d balls in the baskets

so that is the question note one the balls are identical and two none of the boxes will be empty what is the significance of these two

so let me first explain that suppose we have three balls a b c that means we can distinguish between the three balls they are not identical in how many ways we can put in two boxes such that none is empty that is the question therefore we can do it in the following way a comma bc that is one way of arranging b comma ac c comma a b a b comma c ac comma b and bc comma a therefore six possible ways but if the balls are identical then we have only the following possibilities one in the first box and two in the second box and two in the first box and one in the second box there is no other way of arranging this

so only two possible ways why because none of the box will be empty now if we allow some box to be empty then in the first case we can have two more possibilities that is a b c and \emptyset in the other one or \emptyset and a b c

so total eight possibilities in the second case we can have three zero and \emptyset 3 also therefore 4 possibilities i hope you understand the significance of

identical balls and distinguishable balls like a b c and also keeping some of the box empty and keeping no box to be empty that is another variation now our problem is to put ten balls in three boxes such that none of the boxes is empty

so how to do

so consider this 10 balls in among these 10 balls we have to put sum in box one sum in box two and sum in box three

so how many ways you can do the solution to the problem is as follows

so let me start from slightly beginning thing suppose there is only one box then number of possibilities is equal to one you put all the balls in the box suppose there are two boxes then the possibilities are one in the first box nine in the second box two in the first box eight in the second box

so you can easily understand the points are going to be like this

so how many possibilities are there it is very easy that there will be nine possibilities now considered three boxes and none will remain empty therefore in the first box how many elements can be there there can be one two three 4 5 6 7 can we have more than 7 there no because no we can have eight also we can have eight there and then the remaining two we can put one one

so there are eight possibilities for the first box how many balls are remaining now the remaining number of balls are 9 8 7 6 5 four three two

so these can be put in two boxes second and third in how many ways it is very easy to calculate that because when there are 10 boxes 10 balls we found that we can put in two boxes in nine ways therefore when there are nine balls we can put it in eight ways when there are eight we can put it in seven ways and when there are two we can put it in only one way that is one in the second box and one in the third box therefore total number of possibilities is equal to 1 plus 2 plus up to 8 is equal to eight into eight plus one is equal to nine by two is equal to 36 different ways now suppose we have hundred identical balls to be put in 20 different boxes such that none of them is empty then it is not possible to go in this way because at each step as the number of boxes increased we have multiple possibilities coming up

so how to solve it

so suppose there are n identical balls which we have taken ten

so let me draw ten and we need to put it in three boxes

so what we do consider the gap between two balls suppose i put a vertical line there that means i am dividing it into two different parts suppose arbitrarily i choose to such lines then i can say in the first box there is one ball in the second box there are five balls in the third box there are four balls

so this leads to the following configuration one five four on the other hand suppose arbitrarily i choose two lines and these together i will get there are three balls in the first box two balls in the second box and five balls in the third box

so that gives you an idea of how to solve this problem

so out of nine gaps we have to choose two to put the vertical line therefore number of solutions is ${}^{10-1}C_2$ is equal to 9C_2 is equal to $\frac{9 \times 8}{2}$ is equal to 36 we get the same answer that we have got before therefore in general if there are n identical balls to be put in k boxes such that none of the box is empty then the number of possible ways is equal to ${}^{n-1}C_{k-1}$ and i have demonstrated how we arrive at this number

so let us solve a problem based on same principle in how many ways three positive numbers x y z can be chosen such that x plus y plus z is equal to 15.

suppose this is the problem then we can easily break it into a similar the ball and box problem

so consider 15 ones

so we may consider them to be identical ones because they are the same now i have to find out three numbers such that their sum is 15 already i have 15

so i can arbitrarily choose two lines in the fourteen gaps and that gives me the partition two comma eight comma five

so one possible way of arranging them is x is equal to two y is equal to eight and z is equal to five and therefore possible ways is ${}^{14}C_2$ and you can understand how we got this number now consider the following we have ten identical balls to be put in three boxes such that any box remain empty what is the number of possibilities that is the question

so how do we solve it we have the solution for situation when no box can be empty but here any box can remain empty also we do the following

so along with the in balls we also add k
so let me put them in a different color
so this is the n balls and this is the k boxes
so we have n plus k many different items in how many ways n plus k can be distributed into k boxes such that there is no box which is empty
so n plus k elements can be put in k boxes such that none is empty is n plus k minus 1 c k minus 1 and i claim that that is the number of solutions why because now we have k partitions none of them is empty
so how many balls are there n plus k balls right subtract one from each configuration therefore now we will have in balls in k boxes but some box can be empty also
so let me give you a small example say three balls and two boxes we know that solutions are 0 3 1 2 2 1 and 3 0.

so what we are doing we are saying that 3 plus 2 that is five balls are there and i am partitioning it into two boxes such that none of them is empty therefore what we do by the earlier problem out of the four line four gaps we choose one partition

so consider this therefore the arrangement of ball is one four now if we subtract one from each of them we get zero three similarly if we consider this to be the line then we will have the solution to be 2 3 if i subtract 1 from each of them then we get 1 comma 2 now you understand since we can choose the partition is in four ways there are four different arrangements for the balls
so i hope that this concept is clear when the boxes are not allowed to be empty the number of solution is n minus 1 c k minus 1 when the boxes are allowed to be empty then the number of solution is n plus k minus 1 c or choose k minus 1 okay friends i stop here today i hope you understood the concept in the next class i shall start at this point and we shall solve some other problems related to probability okay friends you