

hello students welcome to lectures on complex numbers in the last lecture we discussed problems on n th root of unity in particular cube roots of unity let me recall the last result we discussed in the class that is if t is an equilateral triangle with vertices a b c which is oriented in the anticlockwise direction then it satisfies the following equation that is $a + \omega b + \omega^2 c = 0$ where ω is the cube root of unity and if it is equilateral triangle if and only if it satisfies this equation as well as it satisfies the other equation that is $a^2 + b^2 + c^2 = a b + b c + c a$ let us do a simple problem t be a triangle with vertices let us call it as z_1 z_2 z_3 in the complex numbers with the property that their modulus are equal and their sum is 0 then we have to show that t is an equilateral triangle

so what is given to us is three vertices with equal magnitudes and their sum is zero which is almost equal to the sort of cube root of unity property but what we know is the cube root of unity their modulus

so equal which is one but here it is not said that it is say it is only given that modulus are equal which we do not know that whether it lies on the unit circle or more than the unit circle

so let us try to visualize what is given to us since their modulus are equal so they lie on a circle with some radius we can call it as r

so they are distributed on the circle z_1 z_2 z_3 what is given to us is their sum is zero from this equation that is given factor is the sum of these three complex number zero if we take the conjugation still it is going to satisfy the same equation

so take the conjugation still it is 0 and this implies that it is $\bar{z}_1 + \bar{z}_2 + \bar{z}_3 = 0$ now use the fact that their modulus are equal

so we can divide by divide by $\text{mod } z_1$ square if we divide by $\text{mod } z_1$ square this is the value anyway it is equal to the $\text{mod } z_2$ square as well as $\text{mod } z_3$ square then we get here \bar{z}_1 divided by $\text{mod } z_1$ square which can be written as \bar{z}_1 into \bar{z}_1 plus \bar{z}_2 divided by $\text{mod } z_1$ square which is same as $\text{mod } z_2$ square

so this can be written as \bar{z}_2 into \bar{z}_2 plus \bar{z}_3 divided by $\text{mod } z_3$ into $\text{mod } z_3$ power this is equal to 0 .

so we arrive to the following equation which is $1 + \omega + \omega^2 = 0$ and we can multiply by the factor which is $z_1 z_2 z_3$ as a common multiple for this factor then we arrive to the following equation we get here $z_2 z_3$

so we multiply this equation by the factor which is $z_1 z_2 z_3$ we arrive to the following equation this is equal to zero lets call this as equation one and again from the given equation we have $z_1 + z_2 + z_3 = 0$ then take its square that is again 0 this implies that expanded we get the expression $z_1^2 + z_2^2 + z_3^2 + 2 z_1 z_2 + 2 z_2 z_3 + 2 z_3 z_1 = 0$ and by the equation 1 this factor is 0

so we get that $z_1^2 + z_2^2 + z_3^2 = 0$ let us call it as equation two from equation one and two we see that $z_1^2 + z_2^2 + z_3^2 = -2 z_1 z_2 - 2 z_2 z_3 - 2 z_3 z_1$ now let us recall the proposition which we proved earlier that is if you have a triangle with vertices a b c this is equilateral triangle if and only if it satisfies the equation which is $a^2 + b^2 + c^2 = a b + b c + c a$ from this result we get that t is equilateral triangle hence we have proved the given problem that is if z_1 z_2 z_3 satisfies this

equation then we proved that it is an equilateral triangle let us do another problem which has similar assumption that is we are given three complex numbers z_1, z_2, z_3 such that whose modulus are equal with the value as one and their sum is not equal to zero but their square sum equal to zero that is $z_1^2 + z_2^2 + z_3^2 = 0$ then we have to show that for any integer n greater than or equal to 2 consider the following expression that is $z_1^n + z_2^n + z_3^n$ calculate its magnitude that will be always either 0 or 1 or 2 or 3 ok

so let us try to prove this result

so first observation we will make is that the complex numbers z_1^2, z_2^2, z_3^2 are distinct suppose they are not distinct what happens

so which means that if let us say that $z_1^2 = z_2^2$ then immediately from the assumption we see that $z_3^2 = -2z_1^2$ whose modulus is basically 2 which is contradiction to our assumption that modulus of z_3 is one ok

so we observe that the given complex numbers z_1, z_2, z_3 they are distinct and they lie on the unit circle which means that they are not like lie on a line

so which means we can place a triangle with vertices as z_1, z_2, z_3 and the previous result will tell immediately that it is an equilateral triangle with the vertices z_1^2, z_2^2, z_3^2 let me recall the previous result we showed that if the magnitudes are equal and their sum is equal to zero then we showed that the triangle with vertices as z_1, z_2, z_3 forms an equilateral triangle now we observe that from this we observe that the triangle with vertices as z_1^2, z_2^2, z_3^2 is an equilateral triangle and notice that one more say a note we see that $|z_1^2| = |z_2^2| = |z_3^2| = 1$

so which means they lie on a unit circle and the corresponding vertices gives us gives rise to a equilateral triangle which means that the complex numbers can be connected through the omega that is we can obtain the one vertex by rotating 120 degree that is multiply by the cube root of unity with z_1^2 and we can obtain the other vertex that is z_3^2 multiplied by further say omega to this quantity that is $\omega^2 z_1^2$

so we can notice that omega can be written as ω^4

so which means the same expression can be written as ω^4 multiplied with z_1^2 here we keep as it is then we see that z_2^2 is given by square root of this element which gives us $\pm \omega^2 z_1$ and z_3^2 is $\pm \omega z_1$ now recall what we would like to calculate we would like to calculate the value for $z_1^n + z_2^n + z_3^n$ now calculate the expression that is $z_1^n + z_2^n + z_3^n$ this is equal to we see that z_1^n can be commonly taken out

so we get here $1 + \omega^2 \omega^4 - \omega^2 \omega^4$ again the \pm minus omega power n now we need to calculate its magnitude since modulus of z_1 is one

so modulus of z_1^n is one

so we end up just calculate the magnitude of the following complex number that is $1 + \omega^2 \omega^4 - \omega^2 \omega^4$ thus immediately what we can say is this is less than or equal to three but our intention is to show that the value precisely either 0 or 1 or 2 or 3 okay

so which means that it will not take any other real number other than the integers which is between zero to three

so this can be easily realized because we can see that for any value well this expression will be equal that is ω^{2n} will be equal to ω^n so our question is say when this will be equal this will be equal only when ω^n is one

so it means that if i take n as integer values 2 3 and so on when it is 3 multiple then you get the same value here that is ω^{2n} as well as ω^n you will get the same value which is one so otherwise you get always here which is a different element that is here you may get ω^2 in that case you get here ω one similarly so there won't be any equal element if n is different from multiple of three so with this observation we see that we get only possibility here so possibilities are ω ω^2 we need to sum or ω with minus ω^2

so if i try to again recall i am only focusing on some of this expression so here minus ω ω^2 and other possibility minus ω minus ω^2

so under this possibility you can see that their sum will be always an integer so if i take everywhere the one so their sum one can easily verify that their sum will be only 0 1 2 and 3.

so i leave it as a exercise to make this verification let us do another problem compute the value of the following expression here again ω is the cube root of unity

so we have to calculate the value for plus expression so try to recall the properties of ω one is one plus ω ω^2 this is zero and ω^n if it is multiple of three the power is three or three multi multiple of three then you get one in general we can write it as ω^n as one if n is multiple of three

so which means remainder if i divide by three remainder zero then the value is one ω if n you divide by three if the remainder is one you get the value ω ω^2 if the remainder is two when we divide n by three ok

so now try to see some pattern here that is if i consider the expression one minus ω^2 this is equal to minus ω and remain 1 plus ω^2 is replaced with again minus ω

so we get minus 2 ω if i consider the next expression that is 1 minus ω^2 plus ω^4 which is nothing but ω

so which is one minus ω^2 plus ω the value is again replace one plus ω by minus ω^2

so we get minus two ω^2 if i go the next next expression which is three plus six we see that immediately it is one

so the general observation is one minus ω^n plus ω^{2n} we will get the expression one if n is multiple of three and we get minus 2 ω if n is equal to 1 mod 3 and minus two ω^2 if the remainder term is two when we divide n by three

so with this observation now the product now becomes simple

so now what we can do is that by this observation you can combine three terms consecutively which means that consider the first three terms product

so the first three term product we get which is faster term is minus two ω and the next term is two ω^2 and the next term is one

so the value is two square the remaining is ω^3 that is one if i take the other set of three again you are going to get the two square by this observation the value of the product is given by the following expression suppose n is multiple of three then we can combine three three terms each other and then the product which we are going to get 2 square and the number of times

you are going to multiply as n by 3 this happens when n is multiple of three and when n is say multiple of three plus one then when we combine the three terms then we will left out with one term which will comes from the term which is one minus ω plus ω square

so the expression what you get is two square power which is you divide by 3 take its quotient and then there is a one more factor which we did not combine in three terms which is minus two ω and if n is say $3k + 2$ then with a similar argument we get 2 square this is quotient of n by three multiplied with minus two ω and minus 2 ω square

so now simplifying this expression we get the value for the product factor now let us try to study the geometrical object which is like straight line and then we will go for a circle

so first let us discuss what is a equation of a straight line in the complex plane

so we will show that equation of the straight line in the complex plane is given by $\bar{\alpha} z + \alpha \bar{z} + \beta = 0$ where α is a nonzero complex number and β must be a real number

so let us derive this

so the general equation of a straight line in the cartesian coordinate system is given by $ax + by + c = 0$ where a, b, c are real numbers to represent a line either a or b must be non zero to add this condition $a^2 + b^2 \neq 0$ any pair of x, y satisfies this equation now trace this x, y in the cartesian plane then we get a straight line now for this pair of x, y we can associate a complex number which is namely x is say let z equal to $x + iy$ then the x is given by $\frac{z + \bar{z}}{2}$ and y is given by $\frac{z - \bar{z}}{2i}$ now recall what we have done we are just doing the natural association of pair of element in the cartesian plane to the complex number

so given a pair of element x, y in the cartesian plane we associate the complex number that is $x + iy$ then we are able to see what is this x and what is y in terms of z and \bar{z} now if the pair of element x, y they lie in a straight line then it satisfies this equation

so this implies that we get $a \frac{z + \bar{z}}{2} + b \frac{z - \bar{z}}{2i} + c = 0$ and just say combining the coefficient for \bar{z} we see that it is $a + i \frac{b}{2}$ and z times $a - i \frac{b}{2} + c = 0$ if we consider α as this number then we can simplify further that let say substitute α as $a - i \frac{b}{2}$ then we get $\bar{\alpha} z + \alpha \bar{z} + c = 0$ now since $a^2 + b^2 \neq 0$ that implies that $|\alpha| \neq 0$ which is same as saying that α is a non zero complex number and where c is the another node say observation is

so if i just summarize α is a non zero complex number and c is a real number

so we get the following equation which describes the straight line in the complex plane one can ask what is the slope of this line and if you consider the the slope of the given line which is $ax + by + c = 0$ we get the slope as that is $m = -\frac{a}{b}$ you assume b is nonzero then we could easily derive what is say slope of this equation in terms of α

so slope slope of equation one we need to since α is thus then a can be derived by $\alpha + \bar{\alpha}$ and b is given by $\bar{\alpha} - \alpha$ from this we could see that the slope m is given by $i \frac{\alpha + \bar{\alpha}}{\bar{\alpha} - \alpha}$

so from this standard equation one can easily derive properties of straight lines like when the when the two straight lines will be parallel as well as like to study perpendicular lines this can be done by studying the corresponding

coefficients of the straight line let me give it as an exercise that is we are given two straight lines these two lines are parallel if and only if it satisfies this equation if you recall in the cartesian coordinate system you take these slopes of two lines let us say m_1 is a slope for line one m_2 is a slope for line two if $m_1 = m_2$ then you will say that the two lines are parallel now we are getting similar to that that is $\bar{\alpha}_1$ by α_1 you treat that as some sort of slope in the sort of complex plane if they are equal then you get that it is parallel and similarly we have another expression to describe perpendicular

so we say that two lines are perpendicular if and only if this slope factor which we denote by $\bar{\alpha}_1$ by α_1 that ratio plus the other slope factor their sum must be equal to zero then we say that these two lines are perpendicular

so let us try to study equation of a circle in the complex plane

so what we studied is the simplest one which is the unit circle which is described in the complex plane by $|z| = 1$

so radius is fixed which is one then you trace the points we get center as 0 and radius is 1 and if we try to write as a parametric equation we get $z = e^{i\theta}$ that is $\cos \theta + i \sin \theta$ with θ varies from 0 to 2π and modulus of z is one if i consider similarly center as origin and radius as r which is described by $|z| = r$ and here you get the parameterization as $z = r e^{i\theta}$ where θ is varying from zero to 2π

so in this case we call it as it is a circle with center as origin with radius r now one could ask how to describe a general circle equation in this complex plane

so if we call it as

so here the two cases what we see is that it is specifically centered at origin

so here say center is fixed and whatever we described the circles are only the radii vary if i try to describe any general which means i need to just move the center to some other place ok

so my interest to describe what will be the circle equation if i consider any point in the plane let us say z_0 as center and let us call the radius as r what is the equation for this we have seen such a similar problem that is just say we need to shift this particular one to the origin that is like if i shift by z_0 then the entire points going to move such a way that center is origin with radius as r

so which means what is happening it is simply the general equation of a circle can be realized as just shift the entire say the given circle by z_0 now we have moved this one to the origin and what we described here is if any circle with center as origin then it is given in this parameterization now i have shifted my circle to the origin by z_0 now this circle is parameterized by $r e^{i\theta}$ which is same as saying that modulus of $z - z_0$ is r here θ varies from zero to 2π now let us again look back this equation the general equation of the circle we are writing modulus of $z - z_0$ equal to r now i just squared it this is nothing but with simple notation that is let us say that $z_0 = x_0 + i y_0$ and let us consider the arbitrary point z satisfying this equation as $x + i y$ then modulus of $z - z_0$ square is given by $(x - x_0)^2 + (y - y_0)^2 = r^2$ ok

so from this equation it is clear that from the complex number association to the plane the cartesian plane we see that whatever equation it which describes the general equation of a circle

so let us further examine this equation

so star can be again written as $(z - z_0)(\bar{z} - \bar{z}_0) = r^2$

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so it is each one conjugation that is r^2 and what we get as the expression which is $z \bar{z}$ and other terms are $z \bar{z} + \alpha \bar{\alpha} - z \bar{\alpha} - \alpha \bar{z} + r^2 = 0$.

so now this looks little say viewers general in the sense that we could see that the equation of the form $z \bar{z} - z \bar{\alpha} - \alpha \bar{z} + r^2 = 0$ plus $c = 0$

so which denotes a circle provided with the following condition

so the equation let us call it as say one one describes the circle if what is the condition only we need to focus on c c is nothing but $c = r^2 - |z_0|^2$ square minus r^2 square now just trace back we need to have just this term the r^2 square must be greater than 0 that is the only condition required to get the general circle with center as z_0 with radius as r

so here c is denoted by this expression and we see that r^2 from this r^2 square is given by $|z_0|^2 - c$ thus must be greater than zero that is c is a real number less than $|z_0|^2$ ok

so if i just summarize one denotes the equation of the circle provided the c satisfies this condition ok

so if c satisfies this condition then we could say that this equation describes circle with centered at z_0 but we need to manipulate to get the radius which is r let us discuss a simple problem suppose the set of all complex numbers satisfies this equation for some fixed α and β and a constant k then it represents circle if and only if the distance from α and β whose square must be less than $2k$

so let us try to prove this result suppose a complex number satisfies this equation then $|z - \alpha|^2 + |z - \beta|^2 = k$ then it is same as $(z - \alpha)(\bar{z} - \bar{\alpha}) + (z - \beta)(\bar{z} - \bar{\beta}) = k$ now recall the previous result we will have a we need to to represent a circle we need to get in this particular form

so let us try to simplify in this particular form

so we have $z \bar{z}$ one factor from here another factor from this expression

so we get two times of $z \bar{z}$ and minus $z \bar{\alpha} - \alpha \bar{z}$ and then plus $|\alpha|^2$ and other terms are $z \bar{\beta} - \beta \bar{z} + |\beta|^2$ this is equal to k and now we have $z \bar{z}$

so try to combine the coefficient of z which is $\bar{\alpha} + \bar{\beta}$ and let us collect the all constants $|\alpha|^2 + |\beta|^2 - k$ this is equal to zero this is our constant like c which appeared in the previous one now the center point is nothing but $\alpha + \beta$ by 2

so we see that $z \bar{z} - z(\bar{\alpha} + \bar{\beta}) + \frac{|\alpha + \beta|^2}{2} - k = 0$ plus $\frac{|\alpha + \beta|^2}{2}$ and let us call it as constant c now this will represent a circle if and only if

so this describes circle if and only if our constant c must be less than modulus of z_0 square where z_0 here given by $\frac{\alpha + \beta}{2}$

so which is $\frac{|\alpha + \beta|^2}{2}$ whole square

so we have

so c is less than modulus of $\frac{\alpha + \beta}{2}$ the whole square recall what is $c = |z_0|^2 - r^2 = \frac{|\alpha + \beta|^2}{2} - k$ and we divided by two for the all expression

so c is thus and then this is less than one by four $|\alpha|^2 + |\beta|^2 - k$ square plus $\frac{|\alpha + \beta|^2}{2}$ now simplify the this inequality

then we get minus k less than we have say twice of $\alpha^2 \text{ mod } \alpha^2$ plus twice of $\text{mod } \beta^2$ square once we bring to the right hand side we get that minus $\text{mod } \alpha^2$ minus $\text{mod } \beta^2$ square and plus $\alpha \bar{\beta}$ plus $\alpha \bar{\beta}$ now multiply by the minus sign for both side we get the reverse inequality that is modulus of $\alpha - \beta$ the whole square which is less than one mistake this is two less than two k

so this concludes that the desired result whenever z satisfies this equation and the constants α, β, k satisfies this then we get a circle and whenever we get a circle by this particular equation then the α, β, k should satisfy this condition let me give you some simple exercises suppose z satisfies this equation where α and β are complex number and k is a positive real number which is different from one then we can show that it represents a circle and one more exercise suppose a complex number α which lies in the circle just notice that it is a circle centered at x naught y naught with radius as r and α lies on the circle and suppose $\alpha \bar{\alpha}$ with one by $\alpha \bar{\alpha}$ lies on the other circle but center is same but radius is four r^2 then together with the condition that modulus of z naught square is given by r^2 plus two by two

so then we can determine the value of α

so i will write down the answer

so pl please verify it let me summarize we first introduced a complex number system where we introduce the plus operation as well as the product operation after that we introduce the what is the modulus of a complex number and conjugation of a complex number and then we studied several inequalities after that we studied n th root of unity and several properties and problems based on that in particular we study the cube root of unity and several geometrical problems based on that and the last in this lecture we discuss the several geometrical objects like straight line circles how it can be represented in the complex plane and this gives a sort of illustration that any geometrical problem can be realized in the complex plane and the problems in the complex plane can be transferred to a geometrical problem and this gives some sort of a freedom to solve a problem in a different view point

so with this we end our lectures on complex numbers thank you you