

hello students welcome to seventh lecture on complex numbers in the last lecture we discussed the  $n$ th root of unity and based on that we have proved several identities let us continue on this discussion we have proved the one of the identity which is  $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \dots \sin \frac{(n-1)\pi}{n}$  value is  $\frac{1}{2^{n-1}}$  let us see that how this helps us to solve certain problems find the value of the following expression the expression is  $\sin 10^\circ \sin 20^\circ \sin 30^\circ \dots \sin 80^\circ$  we would like to find the product of this expression

so let us try to use the above identity to find the value of this expression what we observe is given any  $n$  value let us say  $n$  equal to 2 then what we see is that this is  $\sin \frac{\pi}{2}$  and if I take  $n$  value let's say 10 then it goes still say the first value will be  $\sin \frac{\pi}{10}$  which is  $\sin 18^\circ$  and till it goes  $\sin \frac{9\pi}{10}$  by 10 ok

so if I try to identify with the first term where it starts with  $\sin 10^\circ$  very naturally I see that if I take  $n$  as 18 then I am able to get the factor the first factor which is  $\sin 10^\circ$

so which means let us try to start with  $n$  equal to 18 then I am able to coincide with the part of the expression from the above identity

so if I consider  $n$  equal to 18 what do I get here it is  $\sin 10^\circ$  that is  $\sin \frac{\pi}{18}$  and other which is  $\sin 20^\circ$  which is  $\sin \frac{2\pi}{18}$  and it goes till  $\sin 170^\circ$  which is  $\sin \frac{17\pi}{18}$  whose value is we have  $\frac{1}{2^{17}}$  let us see whether we can further simplify it the first observation is if we take the  $k$  value here nine or the ninth term here which is say here if I take  $k$  equal to 9 where my expression is above general term is  $\sin \frac{k\pi}{18}$  I take  $k$  equal to 9 we get  $\sin \frac{\pi}{2}$  which is 1.

so it means that the ninth term in this expression ninth term value is one but the other observation is this is observation one and second observation is if we consider  $\sin 180^\circ - k^\circ$  which is same as  $\sin k^\circ$  in fact you have a general observation that is  $\sin(n-k)^\circ$  this is same as  $\sin k^\circ$   $k$  value from 1 to  $n$  we have the given say expression we have  $\sin \frac{\pi}{18} \sin \frac{2\pi}{18} \dots \sin \frac{17\pi}{18}$  which is  $\frac{1}{2^{17}}$  we observe the ninth term is one and remaining other terms are sort of repeating in the sense that  $\sin 170^\circ$  by 18 which is same as  $\sin 10^\circ$

so we get the square  $\sin^2 \frac{2\pi}{18} \sin^2 \frac{4\pi}{18} \dots \sin^2 \frac{8\pi}{18}$  which is say we can cancel the common term which is  $\frac{1}{2^{16}}$  now this term is nothing but our required expression that is  $\sin 10^\circ \sin 20^\circ \dots \sin 80^\circ$  and whole square is thus and now you take this square root we get  $\frac{1}{2^8}$  as a value now similar to the identity the sign products we get  $\frac{1}{2^{n-1}}$  similar to this we can get a identity involving cos term which is  $\cos \frac{\pi}{n} \cos \frac{2\pi}{n} \dots \cos \frac{(n-1)\pi}{n}$  which is equal to square root of  $\frac{1}{2^{n-1}}$

so now how to prove this identity if I recall when we proved the identity involving sign term what we used is that the polynomial  $z^n - 1$  till this plus 1 this polynomial is factorized using the  $n$ th root of unity and we have considered value of  $z$  as 1 then we got this expression then once you take its modulus value then we got the sign expression now the hinters use say  $z$  equal to  $-1$  in this identity and do the similar procedure

so when you do the similar procedure when you are calculating the distance from  $1 - \alpha^k$  which is same as  $1 + \alpha^k$  modulus where you will get the cast term on the right hand side using this hint you can show that the product term here involving cosine gives this value and where  $n$  is a even number if  $n$  is odd you get this expression and combining the sine and

cosine identities you can derive the identity involving the tangent term ok

so i leave the proof as a exercise now we will discuss the nth root of unity with the particular case n equal to 3 which is called cube root of unity

so we consider n as 3 and we are asking what are all the complex number whose third power is 1 we know that we get the term which is one says angle is two pi by three and other term which is four pi by three the complex number which satisfies this equation are these numbers ok

so there is a special notation which is omega which is called the cis two pi by three we know what is the value of cos two pi by three and plus i sine two pi by three we get the value as minus half plus i root 3 by 2 and what we see is that the term the next term is nothing but omega square that is the cis 4 pi by 3 is just obtained as says 2 pi by 3 which is nothing but our omega square

so using this notation omega what we observe is one omega omega square r cube roots of unity very easily we can see that one plus omega plus omega square the value is zero because you can also observe that omega square is nothing but omega conjugation now it is once you sum omega omega square the imaginary part cancels the real part of omega is minus half

so you will get two times of real part which is minus one

so the sum is zero

so we see that this remark one and similar to the say what we have done in the nth root of unity omega power say n where n is 3 which is we get it as just one ok

so one can just calculate raise its power three which is here for example cis two pi by three cube which is cis three basically goes in the multiplication in the argument you get only two pi that is one ok or we can also see the geometric interpretation for this let us consider the unit circle 1 and omega is placed by say rotating 120 degree and omega square again you rotate by one twenty degree which is in fact the conjugation which is omega square it means that multiplying by omega which is same as that adding 120 degree angle to this particular vector now you add one more 120 degree that is multiplied by omega you reach back one

so we see that omega cube is one and in general if you consider omega power say 3 power n this is same as omega power 3 power n this is 1 n being can be any integer

so the observation is if we have omega power three multiples it is one and one more observation what we made is sum of cube roots of unity is zero like in general we have nth root of unity you sum it up you get zero let us do a simple problem find the value of or simplify this expression in a way that it reduced to the form a plus i b the expression what we see is that this term is nothing but our omega

so using the property of omega we can simplify this easily

so consider this term four plus five omega power three hundred thirty four plus three times omega power three hundred sixty five as we noticed before this term can be written as omega power 3 multiples which is 300 thirty three multiplied with omega plus three times again we can split the power such a way that one is three multiple which is three hundred sixty three this is omega square

so this term is one again this factor is one

so what we get is this is four plus five omega plus three omega square now again use the fact that the sum of cube roots of unity is 0 from this we get omega square is minus omega minus 1 now substitute in this equation we get it as say plus five omega and then minus three omega minus three we get one plus two omega now just see that what is the value of omega which is minus half plus i three root three by two which is after simplifying we see that here we get a minus one here and cancels with one

so the remaining factor is i times root three

so it seems that when we have say expression comes with the powers of  $\omega$  we could easily reduce by using the properties of  $\omega$  let us do a one more problem now we need to show that the power of this complex number which is raised by mul three multiplied with two  $n$  plus 1 which is always minus 1 for every  $n$  being an integer

so let us try to see what is this expression first consider this expression  $\sqrt{3} + i$  by  $\sqrt{3} - i$  let us simplify directly which means you multiply by its conjugate  $\sqrt{3} + i$   $\sqrt{3} + i$  we get here which is  $\sqrt{3}^2 + i^2$  the whole square divided by this is three plus one we get a term which is four now recall the value of  $\omega$   $\omega$  is  $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$  now we see that this expression almost close to this this is equal to  $\sqrt{3}$  we can take out  $i$  commonly if you take out  $i$  outside which is  $i^2$  we can take four inside the square term which means that we get here one minus by two now this is same as this is minus one and this is nothing but minus  $\omega$  which is we see it as  $\omega$  square

so now if we raise the power three times  $\sqrt{3} + i$  by  $\sqrt{3} - i$  three times two  $n$  plus one and what we see is that here minus  $\omega$  square power the three multiple two  $n$  plus one this is minus one power six  $n$  plus three and  $\omega$  square we can write it as six power two  $n$  plus one this gives us minus one and this gives us just one because it is multiple of three which is always one

so we are able to verify our statement let us do a one more problem we would like to find least positive integer  $n$  such that it satisfies this equation

so there may be several  $n$  will satisfy this identity but we need to give the least value of  $n$  for which this equation satisfies

so by this expression we can use that one plus  $\omega$  square  $s$  minus  $\omega$  and  $\omega$  power four which is just say  $\omega$  cube multiplied with  $\omega$  which is just  $\omega$

so which means that this expression we get it as  $1 + \omega$  square power  $n$  plus thus this satisfies if and only if this is minus  $\omega$  power  $n$  and this term is  $1 + \omega$  which is minus  $\omega$  square power  $n$  and this satisfies if and only if  $\omega$  power  $n$  as  $\omega$  power  $2n$  thus satisfies if and only if  $\omega$  power  $n$  equal to 1 we just cancel the  $\omega$  power  $n$  we are asking what is the least value of  $n$  which satisfies this which is simply  $n$  is three let us try to discuss a problem which is slightly different we would like to ask what are all the complex numbers which satisfies this equation

so the moment we see this if we take  $z$  equal to zero one is satisfied which is a trivial solution now let us assume  $z$  is non-zero and we ask whether any non zero complex number satisfies this equation non zero once it is non zero it has inverse or even the modulus of  $z$  which is non zero we can divide by  $\text{mod } z$  square throughout this expression

so we get  $z^2$  by  $\text{mod } z$  square plus  $z$  by  $\text{mod } z$  say here one  $\text{mod } z$  cancelled and we get one this is equal to zero now somehow it is now comes closer to familiar equation which is  $z^2 + z + 1 = 0$  the whole square and another complex number plus one equal to zero which is exactly the  $\omega$  where it satisfies this equation

so let us now ask our question again we are looking for a non-zero complex number which satisfies this equation

so if  $i$  consider a number  $\omega$  as  $z$  by thus then the  $\omega$  should satisfy this equation and we know that there are two unique solution to this equation which is nothing but the cube root of unity which we called as  $\omega$

so and  $\omega$  square

so it has two solutions which is  $\text{cis } \frac{2\pi}{3}$  and  $\text{cis } \frac{4\pi}{3}$  which we call it as  $\omega$  and  $\omega$  square now if we consider  $z$  as  $\lambda$  times  $\omega$  where

lambda is a

so what do you get from this equation

so we get we can choose  $z$  by mod lambda as omega it satisfies this equation

so we just put the value for  $z$  by mod  $z$  which is omega where mod  $z$  can be arbitrary chosen now

so which means  $z$  can be written as lambda times omega and other say lambda times omega square  $r$  where lambda is non negative

so our solutions

so into one let me again repeat what we have done first we observed that equal to zero is a solution and then we look for a non zero solution once we assume  $z$  is non zero we are able to divide by mod  $z$  which means our equation scaled onto unit circle because whatever complex number which satisfies this equation whose modulus is one ok

so we are looking for a complex number which lies on the unit circle and satisfies this equation and that is nothing but the cube root of unity and from that we derived all other solutions now let us try to discuss on the advantage of this complex numbers in terms of geometrical aspect

so the cube roots of unity it has a geometrical interpretation that is when we take the unit circle we place 1 here and omega with 120 degree angle to the positive real axis and other you rotate by the same vector by 120 degree we get omega square what we know is if we place a polygon with the vertex as this cube roots of unity here polygon with three sides which is nothing but the triangle which is regular here regular it means that we are getting a equilateral triangle

so the equilateral triangle let us try to recall what is equilateral triangle

so triangle with all sides are equal that is a say let us call it as this is the length of the side is  $a$  and the other length is  $b$  and thus if it is equilateral then all sides are equal and also all angles are equal not only that we observe here that this equilateral triangle the centroid coincides with the ortho center as well as circumcenter

so we can list several properties for the equilateral triangle for the time being let us just take one of the definition that is all sides are equal and all angles are equal any one of them you can take as the definition these two are equivalent to say that a triangle is equilateral triangle now it is very easy to verify that the triangle which we obtain here is an equilateral triangle simply by seeing what is the sides length sides length are that is one minus omega you could see that this is same as omega minus omega square why because this can be written as omega can be taken commonly and what you get is omega product with 1 minus omega and the modulus can be taken for each factor mod omega is one

so you get that this is one minus omega which means that the side and this side length is equal similarly you can see that this is also equal to 1 minus omega square which is just you can multiply by mod omega then you will see that immediately it is equal to the other side okay

so we could directly verify that the triangle placed with vertices as one omega omega square gives us equilateral triangle now using this property of cube root of unity we will prove some characterization of equilateral triangle we prove the following that is a triangle  $t$  with vertices are denoted by  $a$   $b$   $c$  where the  $abc$  are in complex numbers and we say that  $t$  is equilateral triangle if sat it satisfies any one of the condition otherwise if any of the one condition satisfied then  $t$  is equilateral triangle let us read what is this condition a plus omega times b plus omega square multiplied with c their sum is zero and the other equation is  $a^2$  plus  $b^2$  plus  $c^2$  that is equal to  $a$   $b$  plus  $b$   $c$  plus  $c$   $a$  if this equation satisfied then we can claim that the corresponding triangle with vertices as  $abc$  is an equilateral triangle

so let us try to prove this result first let me prove that if  $t$  is equilateral triangle then we show that it satisfies this equation and similarly if the equation satisfied then we can claim that it is equilateral triangle

so we will show that 1 and 2 are equivalent statements 1 and 2 are equivalent statements

so what is given to us we are given with a triangle which is placed in the organ plane somewhere with vertices are  $a, b, c$  with the orientation is one need to notice the orientation in the anticlockwise direction now when it is given that it is equilateral triangle what we know is the angles are equal are we could also see that the length of each side they are equal now i am going to observe that is it really a like can i study the triangle by shifting to some other point ok the same result can i shift this triangle to some other point and solve the problem there and can i come back ok which is naturally called as shifting property if i

so what is our claim now first we assume that it is an equilateral triangle then we show that the second part what is the second part this is  $a + \omega b + \omega^2 c = 0$  we would like to show this

so if i shift this triangle it means that i am going to add by some complex number to all points otherwise you subtract by a point by each points in this particular triangle meaning us just consider that let us say that you have a triangle with vertices as let us say that it is at  $1, \omega, \omega^2$  and lets say this is  $3, 1$  and let us say this is  $2, 1$  and half ok

so what i could do is i can subtract by  $1, \omega, \omega^2$  then the entire thing can be shifted to the point origin ok

so you subtract  $1, \omega, \omega^2$  to every point in this triangle

so which means i can create a new triangle here without changing any orientation and as well as the side length

so without changing any geometrical property we can just shift which means we can just taking a triangle and place it in the other place ok

so what i can see is that if i shift will it be still the this equation whether it will be invariant ok that is suppose the equation is satisfied suppose say suppose one is satisfied ok then will it be same as true for a other triangle which was replaced to some other point ok which means my new point is  $a - z, b - z, c - z$  they create a new triangle which was just shifted by  $z$  okay now i am asking whether the equation will be satisfied just try it substitute in the equation for this particular point as vertices

so which means it is  $a - z + \omega(b - z) + \omega^2(c - z) = 0$  which is same as  $a + \omega b + \omega^2 c - z(1 + \omega + \omega^2) = 0$  as a common factor  $1 + \omega + \omega^2$  which is zero and since the equation satisfied for  $abc$  we see that this is zero

so now what we observe is if  $a, b, c$  satisfies one then you can shift it to any other place by just subtracting by  $z$  again it will satisfy this equation on the other hand

so it is a vice versa if this particular one is satisfies equal to zero you can actually show that the one is satisfied

so it is like this is call it as two

so what we observe is one and two are equivalent ok from this observation what we are going to do is we are going to without loss of generality we are going to shift our triangle to a origin as centroid to the triangle it means that i am going to shift my triangle such that origin as the centroid what we know the property of equilateral triangle is origin say the centroid and circumcenter are equal

so what we know is

so we have actually shifted let us call it as it is a notation abuse but just

it is a the property is invariant under this shift i call it as again this is a b c now what we observe the angle between these each vertices which makes 120 because this angle which is 60 they are equal angle because of equilateral triangle and since the center is what we see is that it is centroid as well as the circum center we see that it bisects this angle is this particular line bisects our original angle

so which means this is 30 and the other one is 30

so remaining will be 120

so from this observation what we can see is that this side vector which is  $b - a$  if you rotate by 120 degree angle

so if you rotate 120 degree angle it will reach the side which is  $c - b$

so now this observation gives us that in fact it will be equilateral triangle if and only if if my side  $c - b$  is achieved by rotating  $b - a$  by 120 degree angle

so which means that  $t$  is the triangle  $t$  is equilateral if and only if if we get our side  $c - b$  by rotating the rotation is nothing but multiplying by  $\omega$  thus

so this is same as now we see that by simplifying the equation which is  $c + a\omega - b(1 + \omega)$  this is equal to zero and this is same as  $a\omega + b\omega^2 + c$  this is equal to zero now to get our equation just multiplied by  $\omega^2$  then we get here it is  $\omega^3$  that is one here you will get  $\omega^4$  that is  $\omega$  thus

so this concludes that the first remark we are able to achieve that  $t$  is equilateral triangle if one only if it satisfies this condition now we would like to prove the other one that is  $t$  is equilateral triangle if and only if thus

so once  $t$  is equilateral which means automatically it satisfies this equation

so we will just show that these two equations are equivalent

so let us prove that two is equivalent to three or what we are basically like equilateral triangle if and only if the condition three is satisfied again one can notice that if the vertices again the shifting shift property again holds that is if you shift  $a, b, c$  that is a triangle by  $z$

so if this is satisfied if one only if  $b - z$  square plus  $c - z$  whole square this will be equal to  $a - z$   $b - z$  plus  $b - z$  product with  $c - z$  plus  $c - z$  plus

so now again what we are going to do is we are going to make a slight convenience

so that our equation looks very simple simplest choice we will just make the change by  $z$  equal to  $a$  which means shift the triangle  $t$  such that one of the vertex as zero ok

so which means without loss of generality we can assume zero  $b, c, r$  vertices of  $t$  now which means that it satisfies the equation which is given to us that is  $b^2 + c^2 = b c$  now our claim what is the claim we would like to make claim is this is equivalent to satisfying the condition two let me recall what is the condition two

so if the vertices are  $a, b, c$  then this equation satisfies now our vertex  $a$  is zero

so which means we reduce to the equation this is equivalent to saying that  $\omega b + \omega^2 c$  this must be equal to zero now ask what is this mean this is equivalent to saying that we need to say that here just observe that  $\omega - b$  by  $c$  to show that  $b^2 + c^2 = b c$  this is a fun one leaf now it boils down to showing that  $\omega - b$  by  $c$  is cube root of unity now this equation what it gives us is  $b$  by  $c$  plus  $c$  by  $b$  this is one from this we have an expression for  $b$  by  $c$  this implies that  $b$  by  $c$  is one

minus  $c$  by  $b$  now we can try to claim about  $b$  by  $c$  is cube root of unity  
so for this

so our claim is minus  $b$  by  $c$  cube is one

so for this start with minus  $b$  by  $c$  square which is same as  $b$  by  $c$  product with  $b$  by  $c$  but  $b$  by  $c$  is  $1 - c$  by  $b$  now notice that this we get it as  $b$  by  $c$  minus  $1$  now again go back to the expression  $b$  by  $c$  minus  $1$  s minus  $c$  by  $b$  now it is clear that  $y$  minus  $b$  by  $c$  cube is one

so minus  $b$  by  $c$  square multiplied with minus  $b$  by  $c$  the faster term we got it as minus  $c$  by  $b$  and product with minus  $b$  by  $c$  which is one hence we concluded that minus  $b$  by  $c$  as cube root of unity hence we are able to show that the following equation satisfies that is the triangle with origin as one vertex other vertex  $b$   $c$  it is equilateral triangle if and only if these two equations are satisfies but we know that once this equation is satisfies then the triangle must be equilateral triangle which we claimed before

so hence we proved that the following assertion  $t$  is equilateral triangle if on one leaf it satisfies any one of the condition

so in this lecture we discussed several properties of cube root of unity and we discussed several problems based on the cube root of unity and we discuss some more problems in the next lecture thank you you