

hello students welcome to lectures on complex numbers in the last lecture we discussed the n th root of unity and based on that we have proved several identities let us continue on this discussion

so let me recall the n th root of unity it is a complex number satisfying the equation $z^n = 1$ and what we showed in the last classes there are n distinct complex numbers satisfying this equation which is namely given as z^k which is given by $\cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$ for k from 0 to $n-1$ and we notice that these z^k is nothing but z to the power k

so you consider k equal to one which is z that is $\cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$ now take its power k which generates the other n th root of unity and for the convenience we will introduce the notation $\text{cis } \theta$ which is defined as $\cos \theta + i \sin \theta$ and if we take its power k just by de Moivre's law we see that this comes as $\text{cis } k\theta$ and let us make some remarks what we observe is if we consider say put α as z that is $\text{cis } \frac{2\pi}{n}$ if we raise the power n for α what we get is $\text{cis } 2\pi$ by above rule we see that $\text{cis } 2\pi$ which is nothing but $\cos 2\pi + i \sin 2\pi$ which is one and if we raise the power α power k this can be written as α^k and by above observation this is again one for k s greater than or equal to zero similarly one can observe for k negative value say for k less than zero if we consider α^k and power n and one can verify that this is again same as α^{kn} which is again one

so as a summarizing the rule is if we consider the α^k multiples of n that is kn this is one for all k belongs to integer and second remark let us write down these n th root of unity which can be written as α^k with k from zero to $n-1$ our n th roots of unity which means that it satisfies so it satisfies any α^k satisfies the equation $z^n = 1$ will give us one which is same as saying that α^k is root of the polynomial $z^n - 1$ for k zero one till $n-1$ what we know is for any given n one is always root for this polynomial which can be easily seen consider $z^n - 1$ this can be factorized as $(z - 1)$ product with $z^{n-1} + z^{n-2} + \dots + z + 1$.

so by easy calculation we can verify that left hand side that is $z^n - 1$ same as writing in terms of product of these two factors now since $z = 1$ is a root for this polynomial we have factored out the term and remaining factors which is the $n-1$ degree polynomial for this the α^k from k from one to $n-1$ will be root of this polynomial that is from the other observation we can say that α^k is root of $z^{n-1} + z^{n-2} + \dots + z + 1$ for k from one to $n-1$

so it means that we can write the polynomial $z^n - 1$ plus $z^{n-2} + \dots + z + 1$ can be written as $(z - \alpha)$ and so on $(z - \alpha^{n-1})$

so if i write in a short notation what we see is that the factor or the polynomials at power $n-1$ plus z^{n-2} till this plus $z + 1$ this polynomial can be factorized as using the n th root of unity power k and the equality is easily argued because $n-1$ degree polynomial can have at most $n-1$ roots and what we observed is $n-1$ roots we already observed that is the distinct n th root of unity the roots for these polynomials hence this polynomial can be factorized in this form with this remark let us move to a simple problem find the number of ordered pairs (a, b) where a and b are real numbers such that it satisfies this equation $a + ib^{2016} = a - ib$

so we are asked to find possible (a, b) real numbers such that it satisfies

this equation what we know is given ordered pair $a + ib$ given $a + ib$ in \mathbb{R}^2 two cartesian product of real numbers we can associate a complex number $a + ib$ uniquely which means that finding the ordered pair satisfying this equation which is equivalent to find the complex number satisfying this equation because of this relation

so consider z as $a + ib$ and we are looking for all complex number z whose power 2016 should give us \bar{z}

so if we solve this equation for set of all complex number satisfying this equation equivalently we can associate the ordered pairs $a + ib$ satisfying this equation

so let us solve this equation from this equation observation is if we apply the modulus we see that modulus of z^{2016} must be equal to modulus of \bar{z} which is same as $|z|$ which implies that $|z|^{2016} = |z|$ bring to the left hand side

so we have $|z|^{2016} - |z| = 0$

so what we are arguing is if there is a complex number satisfies this equation then it must satisfy this relation this implies that $|z|^{2016} - |z| = 0$ product with $|z|^{-1}$ power 2015 minus 1 this must be equal to 0 from this relation we see that either $|z| = 0$ or $|z|^{2015} = 1$

so we get the relation either $|z| = 0$ or $|z|^{2015} = 1$ if $|z| = 0$ it means that z is zero

so let us consider if z is non zero then $|z|^{2015} = 1$ from this we conclude that $|z| = 1$ why because if $|z| < 1$ then $|z|^{2015} < 1$ and suppose $|z| > 1$ then $|z|^{2015} > 1$ you immediately see that $|z|^{2015} = 1$ cannot be equal to one ok

so by this argument we can see that $|z| = 1$ if its power is one

so from this conclusion

so now consider the case for $|z| = 1$ if $|z| = 1$ then $|z|^2 = 1$ and this will tell us immediately $\bar{z} = 1/z$ we can write $|z|^2 = 1$ as $z \bar{z} = 1$ that is same as $\bar{z} = 1/z$ now under this case we see that we go back to equation from one our complex number should satisfy this $\bar{z} = 1/z$ now we are in a case if $|z| = 1$ then $\bar{z} = 1/z$ is nothing but $1/z$

so this implies that $z^{2017} = 1$ and we know what are all the possible solution for this equation this is nothing but the n th root of unity where n is 2017 .

so it means that the equation $z^{2017} = 1$ has 2017 distinct non-zero solutions and notice that we observed if you consider $z = 0$ also satisfies this equation

so which means $z = 0$ is also solution 1 as a conclusion equation $z^{2017} = 1$ has 2018 distinct solutions let us see a nice property for n th root of unity consider the n th root of unity which is written as z^k given by $\text{cis } \frac{2k\pi}{n}$ where k is zero to $n - 1$ now consider the sum with raising powers to n th root of unities by m summation from zero to $n - 1$ then we get the value as either n or 0 we get n if the power m is multiple of n which is same as saying that n divides m otherwise we get the sum value as 0 let us prove this proposition z^k as we mentioned earlier we can just write it as α^k where $\alpha = \text{cis } \frac{2\pi}{n}$ and k is from zero to $n - 1$ now consider the sum summation k from 0 to $n - 1$ z^k to the power m which is same as k equal to zero to $n - 1$ z^k is replaced by α^k and power m and this is same as summation k from 0 to $n - 1$ α^{km} now this is a fixed number with raised power k now the sum is nothing but geometric sum we

know how to easily find the value of the sum maybe just as a remark i will just add that geometric sum summation k equal to

so let us just recall geometric sum let us say till n minus 1 we have a to the power k the sum value is given as a to the power n minus one divided by a minus one where a must be not equal to one ok

so we can easily derive that the sum value was thus we can apply this here which means that the sum value is here a is the fixed value here α power m

so we get α power m raised its power n minus 1 divided by α power m minus one now let me write the same sum value in the next page summation k equal to zero to n minus one z power k power m its value we found it as α power m power n minus one divided by α power m minus one notice that the α power m will be equal to one if m is divisible by n ok

so which means this formula is valid if α power m not equal to 1 and the equality happens when n divides m

so if n does not divide m which means m is not a multiple of n in that case we know that α power m is not equal to one and the above

so the expression k equal to zero to n minus one z power k power m we see that the term here this is same as α power n power m minus one divided by α power m minus one and notice that α power n value is one

so this becomes zero and this is non-zero quantity when n does not divide m which means we get zero if n does not divide m suppose n divides m which means that m is given by some multiple of n in that case the summation k from 0 to n minus 1 z to the power k we need to raise power m which is q n and we already noticed that n th root of unity its multiple of n power will be always one

so this can be even rewritten as α power k and here it is k q and we easily see that this term is one and again if you raise power k q it is again one the sum value is n if m is multiple of n

so we have proved our proposition that is if you consider the n th root of unities raise power m sum it up all the n th root of unities then you get value n provided m is multiples of n otherwise we get 0 let us see a very particular case with m value as one in particular consider m value as one certainly if n is greater than one it does not divide m

so the sum value from k from zero to n minus 1 z power k where m is considered as 1 we get the sum value as 0.

n is greater than one

so now write down what is this z k z k is nothing but \cos two k π by n plus i \sin 2 k π by n this is equal to 0 and now we can easily see that real part of this complex number is 0 and imaginary part of complex number zero calculate what is the real part of this complex number which is nothing but k equal to zero to n minus one \cos of two k π by n which is real part of this complex number and this is zero and the imaginary part which is \sin 2 k π by n this is 0 if i write down explicitly this is for k equal to zero \cos zero that is one and then the remaining terms what we have here it is two π by n plus \cos four π by n and till \cos 2 n minus 1 π by n the value is minus 1 and we get another trigonometric identity with substituting k values for k equal to 0 we see that the first sum value is 0 the remaining factors we get \sin two π by n \sin four π by n and the last term \sin two n minus one π by n the value is 0.

so for the particular case power m equal to 1 we got a nice trigonometric identity let us do a simple problem based on this proposition we denote α as cis 2 π by n is nothing but the n th root of unity and a complex number z satisfies following condition that is you measure the distance from the n th root of unity α^k to z which is less than or equal to one from all the n th root of unities if there is a z satisfies this condition then the z must be zero

so let us try to see picturically what we see is that if k equal to zero what you have is one say the maximum distance from one to z is less than or equal to one

so we know what is this distance this is unit one the distance also here it is unit one now z lies inside within the distance of 1 from the complex number one okay

so which means you trace the what is the length one from one then we get the circle now z is somewhere inside and now take a another say α which is in the n th root of unity now you make a another circle with the distance one again the z lies inside the circle

so similarly you see that with each n th root of unity the distance to that n th root of unity is always less than or equal to one if such a complex number is there then we are going to show that it is nothing but the origin let us try to prove this result given to us that is the z minus α^k is less than or equal to 1 for all values of k from one to n minus one

so now i just fix a value of k which is in zero to n minus 1 then we see that $|z - \alpha^k|^2$ whose square which is just the again product of each factor and this term is less than or equal to one again this term is again less than or equal to one

so the product is less than or equal to one this implies now try to expand this left hand side this is $z - \alpha^k$ $z - \alpha^k$ bar their product is less than or equal to one now further expand left hand side this is $|z|^2$ and the other terms $z \alpha^k$ bar minus z bar α^k plus you get modulus of α^k square which is less than or equal to one recall that it is n th root of unity which lies on the unit circle

so which means modulus of this α^k value is one this implies that $|z|^2$ less than or equal to $2 \operatorname{Re}(z \alpha^k \text{ bar})$ take all other terms right hand side you get it is $z \alpha^k$ bar plus z bar α^k where k values are from 0 to n minus 1 the same inequality satisfies

so just observe what it means you take k equal to 0 you get the first n th root of unity for that particular complex number this inequality is satisfied k equal to one again the inequality is satisfied with say k equal to one you get α bar plus z bar into α

so we get n inequalities now you sum these n inequalities this implies if i sum this n inequalities left hand side i get n terms of $|z|^2$ this is less than or equal to sum of k from 0 to n minus 1 z multiplied with α^k bar plus this is a common summation we can have it now the you need to apply the proposition which we have just now proved that as the sum of sum of n th root of unities is zero now which is exactly the summation here here we can throw out z as a common factor and then you get on the right hand side let me write down this is less than or equal to z times summation k from 0 to n minus 1 α^k bar plus z bar summation k equal to 0 to n minus 1 α^k and we know that this sum is 0 and here the conjugation can be taken commonly outside to the summation and the sum is again same as thus which is zero which means the right hand side become zero now see that the modulus of z square which is a non negative term which is less than or equal to zero it means that $|z|^2$ must be zero that is same as saying that z is zero

so hence we proved our desired result that if there is a complex number with distance from all n th root of unities which is with the distance less than or equal to one then the complex number must be the origin let us prove another nice result that we are given a n regular polygon let us call it as what vertex is that is p naught p 1 till p n minus 1 which is placed on the unit circle okay

so we are given with the n regular polygon which is placed on a unit circle with this given assumption we are going to show that distance from p naught to p

1 multiply with distance from p naught to p 2 till the distance from p naught to p n minus 1 thus n terms you multiply you get the value n that is interesting identity and moreover using this identity we are going to derive nice trigonometric identity which states that $\sin \pi$ by n $\sin 2 \pi$ by n till the sign n minus 1 π by n gives the value n by 2 power n minus 1 and similarly we have one more identity the difference here we could see that here we have say odd multiple of π divided by 2 n instead of here we have consecutive terms π 2π

so we we have till n minus 1 π by n but here we have odd multiples of π divided by 2 n their product gives us 1 by 2 power n minus 1 which is very interesting identity let us prove this result let us try to see what is the problem which is given to us we have a n regular polygon which is placed in a unit circle its like p naught and p one and

so on p n minus one we know that these vertex can be chosen as n th root of unity ok

so without loss of generality we can assume that the given n regular polygon is placed on unit circle such a way that the vertices are nothing but our n th root of unity and we know that if you place a polygon on n th root of unity as vertices then it is a n regular polygon thus we have already discussed in the last lecture now the first identity which we are proving is you consider the distance from p naught to p one and distance from p naught to p two and throughout for the other vertices distance from p naught we consider and take its product their product gives n this we have to show though geometrically looks complicated but once we write down in terms of complex numbers it is very clear its a easy identity to prove

so without loss of generality we assume that these p ks are placed at α power k that is the n th root of unity k is from 0 to n minus 1.

now recall the identity which we already mentioned as a remark that is our polynomial if you recall this identity z power n minus 1 can be factored as z minus 1 product with z to the power n minus 1 z to the power n minus two and thus polynomial can be factored as z minus α power k where α k is nothing but the n th root of unity

so we observe that this polynomial roots are just the α power k where k is from 1 to n minus 1.

so we are going to recall this identity what we have is z to the power n minus 1 plus z to the power n minus two plus z plus one this is written as product of we have seen as factorized by its roots the roots are here we can say p k or our usual notation α power k now thus identity is true for all complex numbers just choose z equal to 1 then left hand side we have n terms we get n and right hand side we get the product of k from 1 to n minus 1 1 minus α power k we are close to the identity which we are interested to prove now ask yourself what is this quantity 1 minus α power k 1 is our first n th root of unity and k equal to one is the with the angle which is 2π by n we will get here this angle is 2π by n this p one is we have written now α one p one we have written it as α and p two we have written α power two now if i take the absolute value then it gives the distance between p naught which is 1 to p k that is α power k

so we see that if you take the absolute value for this identity we get our desired identity that is the absolute value of this product k from 1 to n minus 1 1 minus α power k and right hand side is a non negative number absolute value gives the the modulus value gives the same modulus of $z_1 z_2$ will give give us $\text{mod } z_1$ product with $\text{mod } z_2$

so which means that this is same as product of k from 1 to n minus 1 product of

the modulus factor $1 - \alpha^k$ which is equal to n and this is nothing but distance from p_0 to p_k this is $p_0 p_1$ product with $p_1 p_2$ till distance between p_{n-1} we get n

so which proves the first identity now we would like to prove the second identity which is $\sin \frac{\pi}{n}$ product with $\sin \frac{2\pi}{n}$ and

so on the product value is n by 2^{n-1} to prove this we need to simply calculate what is the distance from 1 to α^k which is exactly going to get our sine terms let us calculate distance from $1 - \alpha^k$ which is $1 - \cos \frac{2k\pi}{n}$ let us consider its square this $1 - \cos \frac{2k\pi}{n}$ by n minus $i \sin \frac{2k\pi}{n}$ whose mod square we get it as $1 - \cos \frac{2k\pi}{n}$ by n three whole square that is real part square plus imaginary part square $2 \sin^2 \frac{k\pi}{n}$ once we expand this we get a term \cos^2 and we have a sine square term that gives one and we have one more term one here

so together we get $2 - 2 \cos \frac{2k\pi}{n}$ now further you use the identity that is $1 - \cos 2\theta$ can be written as $2 \sin^2 \theta$

so we get this is 2 times of $2 \sin^2 \frac{k\pi}{n}$ now notice what is the value of k k is from 1 to $n - 1$

so we calculated $1 - \alpha^k$ square as four times $\sin^2 \frac{k\pi}{n}$ with k ranges 0 to $n - 1$

so this implies that the distance between p_0 to p_k is given by two times of modulus of $\sin \frac{k\pi}{n}$ since k ranges from 1 to $n - 1$ now you see the argument the argument here the value varies from 0 to π

so the $k\pi$ value $\sin \frac{k\pi}{n}$ to $n - 1$ $\frac{\pi}{n}$ which is certainly less than or equal to π and similarly this is greater than or equal to 0 under this argument sign value is always non negative

so we get the value which is $2 \sin \frac{k\pi}{n}$ which is nothing but the distance from p_0 to p_k now use the previous proved identity which gives $p_0 p_1$ which is $2 \sin \frac{\pi}{n}$ product with $2 \sin \frac{2\pi}{n}$ and product with $2 \sin \frac{(n-1)\pi}{n}$ the product is n hence we see that product of $\sin \frac{\pi}{n}$ $\sin \frac{2\pi}{n}$ till $\sin \frac{(n-1)\pi}{n}$ we get n by 2^{n-1} which is the required identity

so we have proved the second to prove third identity i will just give the hint what we have considered

so let us go back to the identity which we have we would like to prove we get here $\sin \frac{\pi}{2n}$ the previous identity we have $\sin \frac{\pi}{n}$

so to get this particular term we consider two n regular polygons apply this identity

so which means we take number of n being two n ok

so twice of given n you consider that as a regular polygon and then use this identity and with the minimal manipulation we can prove the third identity

so i skip the proof

so in this lecture we saw several propositions based on n th root of unity and some problems we have solved which is on the n th root of unity as well as we saw nice trigonometric identities which we are able to prove using the n th root of unity in the next lecture we will discuss more problems on this thank you