

hello students welcome to lectures on complex numbers in the previous lecture we introduced polar representation of complex numbers let me recall polar representation of complex number given any complex number z can be given by the representation $r \cos \theta + i \sin \theta$ where r denotes the distance from the origin to this complex number and θ denotes the argument which lies between zero and 2π and let me again recall the same number can have different angle because of cos and sin this periodic function with period 2π

so which means the same representation holds for different θ value where the argument again this $\theta + 2k\pi$ where k for some integer let us try to see again a simple example

so when we consider a point let us say complex number $1 + i$ we have the last class we seen that the angle is $\theta = \pi/4$ or 45° and what we observe is you can add say $2k\pi$ it means that from this particular axis we can also view this point as like added with the 360° degree to this line once you rotate by a 360° degree you reach the same place and that is reflected through this say by this periodic functions we see that the same θ can have with the addition with the $2k\pi$ ok

so let us see that one more point which is i considered as its conjugation that is $1 - i$ the angle made from the x axis we see that it is $2\pi - \pi/4$ the angle here it is $2\pi - \pi/4$ and which is same as $7\pi/4$ the same angle can be realized by taking the clockwise direction from the x axis which we denote $-\pi/4$ the reason is you see here when we are measuring the angle to the x axis we are making the angle in the anti clockwise direction that is counter clockwise direction

so that we are taking as a positive way measuring the angle now we are going in the opposite direction

so in this way we are basically denoting the $-\pi/4$ and here if we write with respect to this θ value what we see is $1 - i$ and the representation r denotes the modulus of complex number which is $\sqrt{2}$ \cos of $7\pi/4 + i \sin 7\pi/4$ we can also verify that this is same as with respect to $\theta = -\pi/4$ that is \cos of $-\pi/4 + i \sin$ of $-\pi/4$ this is easily verifiable because you see that \cos function is an even function and use that \sin function as odd function using this immediately we see that this particular quantity is equal to $1 - i$

so the value of we have \cos of $\pi/4$ is $1/\sqrt{2}$ and similarly the sign of now again just repeat the z as the polar representation $r \cos \theta + i \sin \theta$ to make a unique representation we restrict ourselves to θ value zero to 2π and what we observed is it can be with the $\theta + 2k\pi$ with k varying in the integers now what are the say results we discussed in the last class one is the de Moivre formula that is $\cos n\theta + i \sin n\theta$ we showed that \cos of using this formula $\cos \theta + i \sin \theta$ over n $\cos n\theta + i \sin n\theta$ and we saw at the end using this relation we derived some trigonometric identities and now we continue our discussion about the division how we do using this polar representation suppose we have two complex numbers at one in the polar representation $r_1 \cos \theta_1 + i \sin \theta_1$ and $r_2 \cos \theta_2 + i \sin \theta_2$ the division we can let us derive directly when we divide $r_1 \cos \theta_1 + i \sin \theta_1$ by $r_2 \cos \theta_2 + i \sin \theta_2$ here we can get the inverse multiplied by the conjugate in the denominator

so we get the numerator multiplied with the factor this conjugation of $\cos \theta_2 - i \sin \theta_2$ divided by z into \bar{z} is nothing but $|z|^2$ square

so which means what we get is the denominator $\cos^2 \theta_2 + \sin^2 \theta_2$

θ_1^2 which gives one and observe the numerator numerator just focus on the real part $\cos \theta_1$ multiplied with $\cos \theta_2$ and other term comes with the plus $\sin \theta_1 \sin \theta_2$ that is nothing but $\cos(\theta_1 - \theta_2)$

so the real part we get $\cos(\theta_1 - \theta_2)$ plus $i \sin(\theta_1 - \theta_2)$.

so let us recall the multiplication of two complex number what we observed is we multiply the magnitudes that is $r_1 r_2$ and then the angles are summed up now what we observe is angle is subtracted when we are doing the division let us do a simple example consider z_1 as $1 + i$ and z_2 as i let us view in geometrically what these numbers are represent in the organ plane we have the point $1 + i$ somewhere here which makes the angle 45° and i which is making an angle ninety now by division what we observed is we need to divide the its modulus modulus of z_1 is $\sqrt{2}$ and modulus of z_2 is one now

so first of all let us write in the polar form z_1 is $\sqrt{2} \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$ and z_2 is modulus is one and $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ now the division we need to subtract the angle

so this gives us $\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})$

so what we are getting is a vector with the -45° as i mentioned the minus sign denotes that it is measured from the x axis in the clockwise direction

so a vector it goes of this with the 45° in the opposite direction

so again the ah as we noticed previous slide it is $\frac{1}{\sqrt{2}}$ and here it is $\frac{1}{\sqrt{2}}$.

so what you get is which is just the $1 - i$ right

so let me just make a simple remarks here that what we observe is when we divide we observe that it modulus given by $r_1 r_2$ and we know what is r_1 one it is modulus of z_1 and modulus of z_2 and one important point i missed to mention we make the division basically with the z_2 non zero ok this this is always say to then only it is make sense and one more remark which is which extends the demoris formula

so when we are considering z^n we are going to see $\cos \theta + i \sin \theta$ with n being $1, 2, \dots$ thus n equal to -1 it means that excuse me

so that is z^{-1} that is $1/z$ which means that you divide by $\cos \theta + i \sin \theta$ one is also treated as c complex number where angle is just zero which means what you are going to get factor is for this it is we need to say the modulus for each factor is one

so it is just one by one and we need to subtract the angle from the angle for the number 1 ok that is just the $0 - \theta + i \sin 0 - \theta$

so it means it is $\cos(-\theta) + i \sin(-\theta)$ for n equal to -1 we get this and by doing in a similar manner we can verify that z^{-2} this is nothing but $1/z^2$ which is $1/z \cdot 1/z$ we just notice that each factor is comes as $\cos(-\theta) + i \sin(-\theta)$ again its basically the same factor square we know basically by the d morris formula which we derived for positive integer we get it as $\cos(-2\theta) + i \sin(-2\theta)$

so by induction we can show that when n is negative integer we can show that it is $\cos(n\theta) + i \sin(n\theta)$

so as a result combining the positive integers we already verified now we verified for negative integers

so by combining these two we see that for n belongs to integer thus relation

holds that is $\cos \theta + i \sin \theta$ power n gives $\cos n\theta + i \sin n\theta$ now let us do a simple example we would like to say calculate the value for $1 + i\sqrt{3}$ power n plus $1 - i\sqrt{3}$ power n and what we know is the polar representation for this we need to calculate first modulus modulus as two and we need to calculate theta theta we can verify that it is $\pi/3$ so here it is for say \tan^{-1} for thus we have $\pi/3$ so it is \cos of $\pi/3 + i \sin \pi/3$ whole power n similarly we need to do for $1 - i\sqrt{3}$ so again the modulus is same which is 2 and the angle is if you observe it is just in the anti clockwise direction so it is because it is just the mirror image that is a conjugation so it is $1 + i\sqrt{3}$ and this is $1 - i\sqrt{3}$ so if this is $\pi/3$ then this angle is $-\pi/3$ so this is $-\pi/3 + i \sin -\pi/3$ power n by the demoris formula what we observe is the quantity here it is 2^n so the argument n the power n goes with the multiplication in the argument so we get $\cos n\pi/3 + i \sin n\pi/3$ this is first term and the second term 2^n to the power n and use that fact so apply the again this formula we see that this \cos of $-\pi/3 + i \sin -\pi/3$ and after using the fact that \cos is a even function and \sin is a odd function we see that these two factors gets cancelled and we get twice of this term which is $2^n + \cos n\pi/3$ by three so what we observe is when we are calculating power of complex number it seems that using the polar representation we can easily calculate now we are going to discuss something which is very fascinating fact that is the n th root of unity roots of unity so what is the question here we look for complex number whose power is raised by n gives one ok so we are going to ask what are all the complex numbers whose power n gives value one ok so such a complex numbers is called n th root of unity now let us observe so once we demand a complex number satisfies this equation then immediately you observe that modulus of z must be one ok this is first observation we make suppose there is a z in complex numbers such that $z^n = 1$ then immediately we observed modulus of z must be one so from this equation we see that modulus of z^n and modulus of one this must be satisfied and we know that $|z^n| = |z|^n$ and modulus of one is just one and now this is just what we are asking is a real number non negative real number whose power is say raised by n when this will be equal to one this is this happens only when $|z| = 1$ so in the real numbers now with this observation let us try to discuss with the simple cases like taking n equal to very specific values we will try to observe what are the numbers satisfies this equation with fixed value of n let us fix say let us start with just say $n = 1$ ok if $n = 1$ the power basically what we raise just one this must say this implies immediately just it is a single complex number just one ok so now first of all let me just repeat the observation what we made when we are looking for n th root of unity any complex number satisfies this equation then corresponding modulus of say the complex number must be one which means it must lie on the unit circle ok so now i have taken $n = 1$ then it just means that only one point which is $z = 1$ now if i take $n = 2$ then we are demanding say $z^2 = 1$ and simple say observation we can see that this implies $z = 1$ or $z = -1$

equal to plus or minus one

so let us say that first number is one and second number is minus one

so what we observe is when we take n equal to two and we are asking what are all the complex numbers whose square is one we observe that when we consider z equal to one z equal to minus one it satisfies this equation ok

so somehow what we are say observing is it is like given a 360 degree it somehow just divides by two ok

so here it is like 360 degree it just divide by one then only one factor we got now we have power is two then we divided three sixty degree in two factors and which is basically like one factor which is appears at the like angle which is π and another one which is appeared in zero if you again rotate by π we get 360 which is again 1 okay

so now with this observation we are going to just write without thinking for the case n equal to 3.

so we consider n equal to three and our question is what are all the complex numbers whose cube power is one ok

so without say thinking i am just going to apply the previous observation that is we are going to divide this 360 degree angle by three parts

so when i divide which is basically gives something which is 120 as a degree then if i sum up then you get a another term here now i am not sure whether its power is say gives one ok but now you just see that you just see that it is you just try to write the polar representation for this number what is it it is let us call say z one as one z do as the term with the angle which is \cos of since it we have collected a point from modulus one

so modulus of this number is one and angle is 120 degree plus i times sine 120 degree now if you raise the power 3 we get here by the demoris formula the power will be multiplied to the to the argument

so which means the \cos of one twenty degree plus i sine one twenty degree power three which is going to be multiplied for the each argument which gives us three sixty which is \cos of three sixty degree plus i sine three sixty by the morris formula

so we know what is the value for this it is one and this factor is zero

so what we observed is z two cube gives one similarly one can verify z three if you remember how we are apply say how we are getting it is basically angle where equally divided which means you add one more 120 to this particular from this particular line and if you recall the angle sum is happening when you are multiplying it ok

so when say this is basically originally 120 degree comes from z 2 and z 3 is nothing but you multiply from by z to itself now you take the power 3 you see that say each say z^2 cube is one

so we see that it is one

so now let us do the same say observation continue for n equal to four z to the power four equal to one we are asking now what are all the complex numbers whose fourth power is one ok

so again the idea is we are going to apply the same observation that is you are going to divide the angle by four which we get it as π by two

so which is as a default what we observe is when you take z one say ah as one it always satisfies this equation

so which means one is always basically a complex number which satisfies this and further we add the π by 2 which is i and this is minus 1 and minus i and we can easily verify that whose power say four gives one

so here we see that z one is one z two is i z three is minus one z four is minus i ok now whatever we observed till now let us write it as a general frame

so we are considering say our interest to find the n th roots of unity
so which we are say which means that we are interested in a complex number
whose n th power gives one now try to use whatever we have observe now the one
can be written as \cos of $2k\pi$ plus i sine to $k\pi$ ok

so before going to this let me add some more comment to the previous slide what
we observed here is we found like four numbers which satisfies this equation the
question in our mind is whether these are the only whole numbers satisfies those
or more ok this is not answer

so we you need to remember this now z the complex number which you are looking
for let us write it as say $\cos \theta$ and $i \sin \theta$ power n by de Moivre's law
what we know is this is $\cos n\theta$ plus $i \sin n\theta$ which gives $\cos 2k\pi$
plus $i \sin 2k\pi$ where k is from integer ok for simplicity we are going to
just say go from $0, 1, 2$ and

so on

so what we observe is from this we would like to ask what are all the θ values
for which this equation satisfies for some k belongs to the integers

so very naturally what we see is that if θ for θ if we chose as $2k\pi$
by n with k from $0, 1, 2$ and

so on then what we see is that this equation satisfies for these values

so let us call it as equation one for θ equation one holds now

so this particular equation tells us these are the only possible values which
can satisfy this equation ok

so that is the first observation we make

so we are basically like when we are considering a complex number power n must
be equal to one and we try to ask what is the argument for this complex number
we end up that the θ must be of this form now question is what are all the
distinct θ values

so that it gives the distinct complex numbers ok

so here i would like to mention that i use the fact here the observation here
that is $|z|$ must be one ok

so in this equation we used $|z|$ equal to one that is why we basically like
didn't write this fact now our interest to find the distinct θ values from
this particular argument ok

so what i observe is i define θ_k as say $2k\pi/n$ with k from $0, 1$ etcetera
till $n-1$ ok

so i only collect the values for k from 0 to $n-1$ for each θ_k i can
define a complex number

so let and define z_k the corresponding complex numbers which is $\cos \theta_k$
plus $i \sin \theta_k$ now again k is from zero one etcetera this what i know is
from the derivation it follows that these numbers are n th root of unity power n
is one ok for all k from zero one to this $n-1$ now question is whether
these are the only complex number which means can i take k value other than $n-1$
maybe negative or more than $n-1$ if i take we are going to show
that it will be equal to one of the z_k that is what we are going to prove now

so what we will show us our claim here is you collect these sets that is say z_k
in a similar manner you define it from indices for each integer you can
associate a z_k and we are going to show that it is only a finite set which is
just the z_k case with k from $0, 1$ till $n-1$ ok

so to prove this what we need to show is just this set is just contained in
here because it is obvious that this particular finite set of number it is
subset here and immediately we see that the set is contained in here we need to
show that any element here it is of this form okay

so let to prove this let us consider an integer let us call it as let us say r
which is in the integers and we consider z^r which means corresponding θ_r s

θ_r is given by $\frac{2\pi r}{n}$ and correspondingly we have the z^r ok right now we need to observe this factor r ok

so r can be say by quotient remainder theorem that is any say integer can be factored because n is a positive integer given to us with a remainder let us say k where q is a number from integer and k is element in zero to n minus one

so now just apply this θ_r as $\frac{2\pi q}{n} + \frac{2\pi k}{n}$

so the first factor what we observe is it is $\frac{2\pi q}{n} + \frac{2\pi k}{n}$ and we know that if a argument differ by 2π integer multiple we are going to get the same complex number

so which means the number which is the z^r here given by \cos of $\frac{2\pi q}{n} + \frac{2\pi k}{n}$ plus i sine $\frac{2\pi q}{n} + \frac{2\pi k}{n}$ and using the periodicity of \cos and sine function we see that let us $\cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$ by n this is nothing but our element z^k which is associated to the you can write is this as θ_k

so what we showed us any say for any given integer you consider the corresponding argument you define the corresponding complex number this must be equal to one of the z^k

so which means that now what we derived is the n th root of unity is now we can define n roots of unity for that is $z^n = 1$ are that is that case with which is defined by $\cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$ with k values from 0 to $n-1$.

now again just repeat whatever we have done before for example like we take n equal to one i think that is trivial case let us take the n equal to two then we are looking for $z^2 = 1$ then here $z = 1$ is given by

so we by this index notation z^0 which is given by

so here n is two and for the first k value is zero

so $\cos 0$ is one and $\sin 0$ is zero $z = 1$

so what you see is that this is $\cos \pi + i \sin \pi$ we get minus one similarly n equal to three we see that for the equations that cube equal to one we get $z^0 = 1$ that is common one $z^1 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ $z^2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

so $\frac{2\pi}{3}$ which is nothing but the one twenty degree plus $i \sin \frac{2\pi}{3}$ and z^2 which is $\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$ now let us try to visualize this in the geometrically

so when we take the n th root of unity drawing circle is really difficult for me ok just assume imagine that we have this bigger circle now what we observed is when we calculate the n th root of unity take k equal to 1 then it is $\frac{2\pi}{n}$

so we have the whole say angle which is 2π now you segment this by n terms ok

so we are equally segmenting it

so if n is let us say that if n is 8 then we are going to segment the angle by eight

so we get say $\frac{\pi}{4}$ we get $\frac{\pi}{4}$ $\frac{\pi}{2}$ and then further you make a division here and we are getting this one

so if you closely look this then what we are getting is for each segment you are dividing in a equal arc length ok for each division angle because each time what you are having is the angle is $\frac{2\pi}{n}$

so and i when i write θ_1 which is $\frac{2\pi}{n}$ and next angle which is going to add

so θ_2 is say you are going to add this term that is $\frac{2\pi}{n} + \frac{2\pi}{n}$ which means the equal angle we are going to add it ok

so which means if i try to place a object like a polygon on these vertices okay then what are you going to get is a n regular polygon ok

so for example for n equal to 8 if i like to place a polygon then what i get is it is ok it is difficult to visualize from this circle which is drawn badly ok i am ok

so we see that there are eight phase for this polygon and what we see is that each sides are equal and each angles are equal by the definition what we are getting is a regular polygon if we place on the n th root of unity

so let us write down this geometrical observation the geometric image of the n th roots of unity are the vertices of a regular polygon with n sites inscribed in the unit circle with one of the vertices at one this is let us call it as remark one remark two if you look back the definition z one if we look back the definition of z case k equal to zero is a trivial one that is one and let us consider z one z one is \cos two π by n plus i sine two π by n now if you take the power z one square we get by d morris formula that the power multiplies of the argument

so we get \cos four π by n plus i sign four π by n or in general if i take k th power then we get two k π by n plus i sine two k π just by the d morris formula we get this just recall what is this this is nothing but your z k

so what we observe is these that gas that is the remaining n th root of unity it is just this generated by z_1

so if i write down the set the n th root of unit is the n th root of unit is we have set which is z naught z one z two and

so on till z n minus one this is just given by z one power zero that is one power zero one and z one power one we get z one itself z one power two we get z two and z one power n minus one we get z n minus one

so which means the n th root of unity it is just generated by this element z one if you have only the z one we can just calculate the remaining elements just taking its power today's class we introduced n th root of unity and we just discussed that it contains n distinct elements for the n th root of unity and the set is generated by the single element for example the z one generates the remaining all elements

so so we continue the other properties of the n th root of unity in the next class thank you you