

hello students in the last lecture we discussed modulus of complex numbers and some basic inequalities in this lecture we will discuss polar representation of complex number

so let us start with the complex number  $z$  equal to  $x + iy$  where  $x$  and  $y$  from real numbers once we are given with a complex number we know that we can associate this to a ordered pair  $x$  comma  $y$  it means that we can associate this point in the plane let us call this as the point  $z$  which is  $x + iy$  where  $x$  going to denote what is the magnitude in the  $x$  axis when we project this point in the  $x$  axis and similarly the magnitude of this in the vertical direction which is  $y$  now what we observe here is given any complex number we associate a point in the cartesian plane and this is standardly taken as  $x$  axis but in the interpretation of complex numbers in the plane we can treat this as the real axis and here this is imaginary axis and now in this plane for each point we associated a complex number and such a complex number plane is called argon plane or complex plane now given any point in the plane we can also determine this point if we know the distance from the origin let us call it as  $r$  and theta which is angle made to the positive  $x$  axis

so let us call it as this is origin zero comma zero or the complex number zero and let us call this as point  $p$  now what we are saying is  $r$  denotes distance from the origin we are calling the point  $o$  to the point  $p$  and theta is denotes the angle between between the joining line joining segment  $op$  and positive  $x$  axis or the real axis

so if we are given that  $r$  and theta then we can determine what is this  $x$  which is simply write down the relation that is  $\cos$  theta is given by adjacent by hypotenuse

so we get  $x$  as  $r \cos$  theta similarly we can see that  $y$  is  $r \sin$  theta

so what we are observing if the distance and angle is  $r$  given then  $x$  is written by  $x = r \cos$  theta and  $y$  is given by  $y = r \sin$  theta

so it means that our complex number  $z$  can be written as  $z$  equal to  $r \cos$  theta plus  $i r \sin$  theta where  $r$  is the distance from the origin to the point  $z$

so which is non negative and theta can be taken as from zero to two pi and further this can be written as  $r \cos$  theta plus  $i \sin$  theta and such a representation is called polar representation this representation is called the polar representation of  $z$

so again let me just recall if you know the components  $x$  comma  $y$  we are associating a point in the plane and further this point can be just equivalently determined by the distance from the origin and the angle theta

so here we get the factor that is  $r$  and theta which is called polar coordinate system which is just the distance from the origin and the angle and the what we are familiar is cartesian coordinate system which is  $x$  comma  $y$  suppose  $r$  and theta is given

so we start with suppose given  $r$  and theta then we see that we can associate the point in the cartesian coordinate system as  $r \cos$  theta and  $y$  as  $r \sin$  theta conversely suppose we are given with  $x$  comma  $y$  now question is how do we get this  $r$  and theta we can again just go back to this picture by pythagoras theorem we could immediately realize  $r$  is nothing but square root of  $x$  square plus  $y$  square

so given  $x$   $y$  the distance from the origin is straight forward  $r$  is given by square root of  $x$  square plus  $y$  square and theta what we observe is  $\cos$  theta as  $x$  by  $r$  and  $\sin$  theta as  $y$  by  $r$

so we see that the theta should satisfy these two equations simultaneously

so by combining these two equation we see that theta must satisfy this equation that is  $\tan$  theta equal to  $y$  by  $x$  now still like this particular point is not clear that immediately how do we calculate the value of theta if  $x$  and  $y$  is

given

so i will make some simple remarks

so in order to calculate the theta

so before going to that let us let me introduce some notations we write the complex number in the polar representation that is  $r \cos \theta + r i \sin \theta$  which is thus and we can write it as  $r \text{cis } \theta$  where the  $\text{cis } \theta$  is  $\cos \theta + i \sin \theta$  and  $\theta$  is denoted by argument of and of complex number  $z$  and we also observe that  $r$  is nothing but just the modulus of  $z$  which is the observation now some simple remarks what we observe is once we start with the polar representation  $\cos \theta + i \sin \theta$  by periodicity of  $\cos$  and sine function what we observe is this can be written as  $\theta + 2k\pi$  plus  $i$  times  $\theta + 2k\pi$  where  $k$  is in the integers it means that when i take  $r \theta$  or if i take  $\theta + 2k\pi$  they map to one element which is  $x = r \cos \theta$  and  $y = r \sin \theta$  in other words what we observe is if  $\theta$  differs by  $2\pi$  still they map to the stem point which is  $x, y$  how do we understand this by geometrically

so we have a point which gives the  $\theta$  angle from the positive  $x$  axis now the same line can be measured as you go one cycle by  $2\pi$  and then further this  $\theta$  which means this is  $\theta + 2\pi$  again it denotes the same line as angle as  $\theta + 2\pi$  similarly if you take any  $k$  multiples of  $2\pi$  with  $k$  being positive integer we could see that we go around by  $2\pi$  multiples which means we again come to this initial line then we add this angle  $\theta$

so geometrically what we observe is it represents the same line if we take the  $\theta + 2k\pi$  now question is how about the negative values of  $k$  let us see that

so we have a line which makes the angle  $\theta$  now in the observation that this is something which is  $r, \theta$  in the polar coordinate system what we are saying is this is same as  $r, \theta - 2\pi$  i am just considering with the negative value  $k$  that is  $k$  as minus one now this can be seen as i measure the angle in the other direction that is the clockwise direction which makes it minus  $2\pi$  and then i add a angle which is  $\theta$

so in other words what we are making here a convention is when we measure an angle between two lines we fixed our  $x$  axis as  $\theta$  equal to zero and if you measure in the anti clockwise direction  $\theta$  is treated as positive radians and if we measure in the other direction which is the clockwise direction then the angle which is measured in the negative radian let us see whether it is consistent with our representation suppose i start with let us say a point which is  $1 + i$  let us calculate what is the angle and distance distance is square root of  $1 + 1$  which is  $\sqrt{2}$  and  $\theta$  by the relation which is  $\tan \theta = y/x$  here  $y$  is one and  $x$  is unit one

so we see that  $\theta = \pi/4$  and now what we are meaning by minus  $\theta$  is in the opposite direction which means that minus  $\pi/4$  if you measure from the other side which is like minus  $\pi/4$  which is nothing but reflection of this line with respect to the real axis

so if we do the reflection this will give us nothing but the  $\bar{z}$  which is  $1 - i$  but let us see whether it consistent with the polar representation i started  $z$  as thus  $1 + i$  if i write the polar representation  $r$  is  $\sqrt{2}$  and the angle which is  $\pi/4$  now we consider a point let us say consider minus  $\theta$  point that is  $\sqrt{2} \cos$  of minus  $\pi/4$  plus  $i$  sine  $\pi/4$   $\cos$  is a even function and sine is an odd function what do we get as which is  $\sqrt{2} \cos$  which is  $\sqrt{2} \cos \pi/4$  minus  $i$  sine  $\pi/4$  which we get nothing but  $\bar{z}$  ok

so this example illustrates that if we measure angle in the anti clockwise direction we measure in the positive radians and if we measure the angle in the

clockwise direction we measure in the negative radians which is exactly represents like the conjugation that we are able to see with the polar representation as well let us generalize this example

so suppose we have a point in the complex plane  $z$  which is with the angle  $\theta$  now we take its exactly the reflection which is  $\bar{z}$  now we know that it is angle  $\pi - \theta$  but our convention we measure it as  $-\theta$  let us see what is the polar representation we get here it is  $r \cos \theta + i \sin \theta$  by definition taking this as an angle then we get something which is like  $z'$  which is a complex number given by angle  $\pi - \theta$  which is  $r \cos(\pi - \theta) + i \sin(\pi - \theta)$  then immediately we observe that this is nothing but  $\bar{z}$

so this point explains that what is the meaning of taking say  $r \cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)$  with  $k$  varying in the integers with the even the negative numbers we see that it maps to one number in the cartesian coordinate system which is  $r \cos \theta$  and  $y$  as  $r \sin \theta$  now again what we are trying to do our interest is for each point in the complex plane we would like to give a different representation in terms of  $r$  and  $\theta$  now we observe that  $\theta$  can be taken as  $\theta + 2k\pi$  which also represents the same point at the same time to cover this entire plane what we need to do is we need to vary  $\theta$  for two  $\pi$  interval length it means that you take a  $\theta$  with  $\theta$  naught to  $\theta$  naught plus two  $\pi$  where  $\theta$  naught can be any real number if we vary our  $\theta$  in this region then it covers the entire plane therefore what is important is the argument must vary in a  $2\pi$  interval

so the conventional one is you consider  $\theta$  not as  $0$  which we get it as  $0$  to  $2\pi$  if we recall again this it means that we start with the initial segment as positive  $x$  axis where  $\theta$  is zero and then you measure in the positive orientation that is anti clockwise with a rotation of two  $\pi$

so which covers the entire plane and another standard one is  $-\pi$  to  $\pi$  so now here in the entire discussion positive  $x$  axis always as  $\theta$  is zero this particular line is always  $x$  axis always consider  $\theta$  is zero now we know what is the meaning of the positive radians which means you start from zero and then come this cycle which is here this line which is  $\theta$  as  $\pi$  and other region we cover by going clockwise direction

so here this particular line if we measure this will be  $\theta - \pi$  so we observe that the interval length is two  $\pi$  and if we vary our  $\theta$  in this interval then we cover this entire plane and the considering the angle in this region which is called as principal angle or principle argument and of  $z$  which is a  $\theta$  which lies in the interval  $-\pi < \theta \leq \pi$  we would like to ask what is the polar representation for the point origin you consider the origin now as we discussed what we do is we take a line which is starts from  $0$  and pass through the point ok

so if you take any line starts from  $0$  and anyway it pass through  $0$  as well so which means we can take any line which starts from zero and  $r$  is the distance from origin to the point which is itself which means  $r$  is zero but if i take the  $\theta$  any line pass through the zero which we are going to take the as the angle

so which means since the all lines basically from zero whichever pass through it contain zero which means  $\theta$  can be arbitrary here  $\theta$  is can be any it means that if i take  $r$  equal to zero and  $\theta$  can be any value represents the origin it means that we do not have well defined polar representation for the origin

so one need to be careful with respect to the origin point we do not have a well defined representation with respect to the polar coordinate system now we are close to discuss about how to calculate the  $\theta$  value

so to get a unique value for this we restricted our self in the principal argument that is theta as minus pi to pi now there is one more issue that is tan is periodic with period pi

so to get this calculation we need to do some adjustment with respect to the quadrants where x and y lies in formula for principal argument of z first observation is tan is a periodic function with period pi which can be easily verified or in general what we have tan of x plus k pi which is tan x for k in integers if we try to find

so our interest is to find the principal argument such that tan of theta is y by x but by the above relation we see that theta will be given as tan inverse of y by x which going to involve say with the angle plus k pi where k is in the integers

so to avoid this multivalued again we restrict the value of tan inverse to the interval minus pi by 2 to pi by 2

so restrict tan inverse of y by x in the interval minus pi by two to plus pi by two which we call it as arc tan of y by x that is our tan function means that it is inverse function of tan such that the value is restrict to the interval minus pi by two to plus pi by two

so therefore theta is arc tan of y by x and we need to make a proper adjustment by some multiple of pi

so we will discuss now how to adjust some value of k plus such that we get our principal argument of given complex number z

so our principal argument formula that is argument of z z is given by arc tan of y by x plus k plus pi where k plus which is proper adjustment depending on the where our x y lies in

so k plus 0 if x y lies in the first and fourth quadrant which is given by x is positive or y is positive and y is less than or equal to zero

so for both cases we see that the value for k plus we do not need to adjust here because the arc tan inverse going to give value between minus pi by two to plus pi by two which covers the first and fourth quadrant first quadrant and this is fourth quadrant and if x y lies in the second quadrant which is x is negative and y is non-negative we take we add pi to the obtained value r tan of y by x and minus one if x y lies in the third quadrant

so what we are missing here is x equal to zero if x equal to zero then it is the y axis and we know that say if x equal to zero non zero we get this formula if x equal to zero then argument of z is pi by 2 if y is positive negative we take minus pi by 2 let us see a simple examples suppose we are given say minus one i now r is say z is minus 1 plus i then r is given by square root of one plus one which is root two and the principal argument theta is given by arc tan of y by x plus the k plus pi where the point lies in the second quadrant

so we take k value as one

so this is minus pi by four plus pi we get that three pi by four

so similarly you can calculate for other complex numbers which is z dash root 2 plus 2 root 3 i calculate what is its polar representation by calculating r and theta let me summarize the formula for principal argument of z suppose the point lies in first and fourth quadrant then just the r tan of y by x and if it is in the second quadrant arc tan of y by x plus pi if it is in the third quadrant arc tan of y by x minus pi let us do one more example suppose we are given a complex number z as 1 plus cos a plus i sine a where a lies in the interval 2 pi now we would like to calculate or find the polar representation for these complex numbers

so let us just notice that this is not in the polar representation form so to convert

so first calculate  $r$   $r$  is given by square root of  $x^2 + y^2$  which is  $1 + \cos^2 a + \sin^2 a$  which is  $\cos^2 a + \sin^2 a$  we have here

so we get  $2 \cos a$  which is  $2 \cos a$  which is by trigonometric formula we see that this is  $2 \cos^2 a$  by two and we get the  $r$  as  $2 \cos a$  by two  $r = 2 \cos a$  and argument  $\theta$  we will calculate accordingly to the formula

so for  $a$  in the interval  $0$  to  $\pi$  we see that  $z$  is in the first quadrant which means that the  $\theta$  directly the  $\arctan$  of  $y/x$  let us calculate the  $\arctan$  of  $y/x = \sin a / (1 + \cos a)$  which we get it as this is  $2 \sin a / (2 \cos^2 a)$  by two by two  $\arctan$  of  $\tan a / 2$  now notice that since  $a$  is in the interval  $0$  to  $\pi$   $a/2$  is in the interval  $0$  to  $\pi/2$

so the  $\arctan$  gives just as it is  $a/2$  for you the other simple exercises if  $a$  lies in the interval  $\pi/2$  to  $\pi$  just show that or derive that  $\theta = a/2 - \pi/2$  verify that  $\theta = a/2 - \pi/2$  therefore  $z$  is written by  $2 \cos a / 2$  with says the angle which is going to vary says of  $a/2$  if  $a$  lies between  $0$  to  $\pi$  and for other region  $a/2 - \pi/2$

so for the value  $a = \pi$  we can notice that  $z = 1 + \cos \pi + i \sin \pi$  which is  $0$  and  $0$  has no unique polar representation

so with this examples let us move to multiplication of complex number using polar representation suppose we have two complex number  $z_1 = r_1 \text{cis } \theta_1$  and  $z_2 = r_2 \text{cis } \theta_2$  now if we multiply these 2 just by the usual complex number multiplication we can notice that this is  $r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2))$  their product comes as  $\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$  by the trigonometric formula we can see that this factor is  $\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$  this can be written in our notation  $\text{cis } (\theta_1 + \theta_2)$ .

so we made a important observation that the multiplication of two complex numbers become simple if we do in the polar representation that is when we multiply  $z_1 \cdot z_2$  we scale the  $z_2$  by the factor  $r_1$  and we rotate by a angle which is  $\theta_1$  with respect to the origin

so what we observe here as like modulus of the product of two complex number by polar representation we see that the modulus is nothing but  $r_1 r_2$  and again  $r_1$  is nothing but  $\text{mod } z_1$  other factor is  $\text{mod } z_2$

so we see that the identity very easily we are able to realize modulus of  $z_1 \cdot z_2$  is nothing but modulus of  $z_1$  multiplied by modulus of  $z_2$  and the argument of  $z_1 \cdot z_2$

so since we try to consider the principal argument

so it may happen that when we do the sum of two angles it may go more than  $\pi$  or else it can go below then minus  $\pi$

so in order to get the principal argument we need to do some adjustment let us see what is that adjustment need to be made

so argument of  $z_1 \cdot z_2$  when we multiply we get the sum of principle argument of  $z_1$  argument of  $z_2$  and we need to make a adjustment which is  $k \cdot 2\pi$  where  $k$  plus is given by the values if suppose the argument of  $z_1$  and argument of  $z_2$  to sum suppose they lie within the principal argument range then no need to change anything but if it is more than  $\pi$  or less than the minus  $\pi$  say it is less than say minus  $\pi$  then we need to add which means  $k$  plus is one and if it crosses the range then we need to subtract by two  $\pi$  is less than or equal to minus  $\pi$  greater than  $\pi$  let us do a simple example

consider  $z$  one as one plus  $i$   $z$  two as root 3 plus  $i$

so the multiplication we do by converting to the polar coordinates here we know the what is the  $r$  which is root 2 and the angle is  $\pi$  by 4 which we have already calculated and here the modulus is 2 and we can calculate its angle which we can you can verify that it is  $\pi$  by six and then their product becomes just the scale the modulus  $\text{cis}$  and the sum this angles which is four  $\pi$  by twelve d morris formula which states that if  $z$  does  $r \cos \theta$  plus  $i \sin \theta$  then the  $z$  power  $n$  as simply  $r^n \cos n \theta$  plus  $i \sin n \theta$  for  $n$  being greater than equal to one

so the formula is very easily derived from the multiplication what we observed for example if you take  $z$  square which means  $n$  equal to 2 which is  $z$  into  $z$  which says that scale the modulus factor that is  $r$  square and sum the arguments which we get it as  $\cos 2\theta$  plus  $i \sin 2\theta$

so by induction we can verify that  $z$  power  $n$  is nothing but  $r^n \text{cis } n\theta$  multiplied by  $\text{cis } n\theta$

so let us do a simple example which is say  $z$  is one plus  $i$  calculate  $z$  to the power say thousand then by say we would like to calculate this if we do the direct multiplication then computationally may not be easier but we see that if you go to the polar representation which is root 2 and  $\text{cis } \pi$  by 4 and now just the d morris formula says that take the powers to  $r$  which we get it as  $2^{500}$  and then just the  $\text{cis } 1000$  times  $\pi$  by 4 which is say two fifty  $\pi$  and which we know that it is just one one simple exercise for you let us prove the following identity  $\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$  and  $\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$  hint use that  $\text{cis } \phi = \cos \phi + i \sin \phi$  in this lecture we discuss the polar representation of complex number and we observe that the multiplication becomes simple if we use the polar representation and in the next lecture we will discuss the again further results on this you