

in the previous lecture we introduced complex numbers and its algebraic operations addition and multiplication let me start recalling the previous definitions which we have introduced first let me again repeat the definition for complex numbers which we denoting by  $c$  it contains set of all elements  $a + ib$  where  $a$  and  $b$  comes from real numbers and what we said is the two complex numbers are equal that is  $a + ib = c + id$  this is equal if and only if this is again a definition

so if  $a = c$  and  $b = d$  under this say equality relation what we showed is we can identify each complex number to a ordered pair that is  $(a, b)$  from the plane  $\mathbb{R}^2$  let me recall the operations that is addition

so the addition of two complex numbers defined as you sum the real part and the imaginary part and the multiplication which we defined as  $(a + ib)(c + id) = ac - bd + i(ad + bc)$  ok

so with respect to this two operations we mentioned that it satisfies closure law associative law commutative law and identities as well as the inverse they are exist and distributive law satisfies

so with respect to this plus and the multiplication the complex number system becomes a field its a field and what are the other remarks we noticed considering this has the product in the complex number system we observed  $i^2 = -1$  if you remember though we started  $i$  is an imaginary number we identified with the in the plane that is nothing but zero comma one

so which means it is no more an imaginary number we are able to identify the complex number system with the plane any number in the complex number system we are able to associate the element in the plane

so with respect to this particular dot product on this complex field what we observed is  $i^2$  becomes minus 1 and similarly one can verify that if you consider just the points on the real line that can be written as  $a + i0$  that is simply we will denote it as  $a$  dot with  $b + i0$

so under this multiplication we can just see that it is  $(a + i0)(b + i0) = ab + i0$  this is a product in the real numbers plus  $i0$  which we are going to denote just by  $ab$  and if we consider say the imaginary numbers that is say  $i b$  dot product with  $i d$  then we will we can show that with respect to that product it becomes  $-bd + i0$  ok plus  $i0$  this we are going to denote just the real part and with this say multiplication what we can really actually see is that the  $(a + ib)(c + id)$  i can do direct multiplication now how do i define the product

so now i can how can i do it

so here  $a$  is a complex number it is carrying only the real part and  $ib$  is a purely an imaginary number and similarly  $c$  is another complex number but just only with the imaginary part is zero

so we can really see that this is  $z_1$  this is  $z_2$  this is  $z_3$   $z_4$  now this particular product you are going to use the distributive law that is a dot

so the set one multiplied with this  $(c + id) + (ib)$  multiplied with  $(c + id)$  now we see that this is again further say distributive law further one more time you apply this is  $ac + iad + ibc - bd$  all together we just receive what we defined as a product now

so what is the message messages

so let me just see that here we define the product in this manner then the resultant vector is a complex number then we verified all the laws and then using the say properties of this field axioms now we are able to see that

so no more no more say need to remember the product just do the as usual the product and here we are we used the fact that that is  $i^2 = -1$

so using this fact we can basically do the as usual product okay now let me just introduce some more notations one is say the real part of

so let us just say that  $z$  is  $a + ib$  real part of  $z$  which is defined as  $\operatorname{Re} z$  which is  $a$

so real part of  $z$  is  $a$  and imaginary part of  $z$  which is use the notation  $\operatorname{Im} z$  which is  $b$  and if say let us say this is one two third if suppose the real part of  $z$  is zero then we say that we say that  $z$  is purely an imaginary number

so either we say it is purely an imagined number or just an is an imaginary number similarly if imaginary part of  $z$  is zero then we say that  $z$  is purely real

so with this simple notations let me start with the new definition that is conjugate of a complex numbers

so conjugate of complex number is defined by  $\bar{z}$  conjugate of  $z$  where  $z$  is  $a + ib$  where the notation used as  $\bar{z}$  which is given by  $a - ib$  ok

so this is definition for  $\bar{z}$  let us see what is means

so we have the complex plane and let us consider the point  $z$  that is  $a + ib$  so which means in the imaginary unit it is  $ib$  and on the real unit it is  $a$  now  $\bar{z}$  denotes exactly we are taking the unit which is minus  $ib$  and  $\bar{z}$  we are denoting as  $a - ib$  if we see this this is nothing but mirror image of  $z$  with respect to the real axis

so  $\bar{z}$  is nothing but is mirror image of  $z$  with respect to we can say  $x$  axis or the real line let us see simple example suppose  $z$  is let us say that two plus three  $i$   $\bar{z}$  is just two minus three  $i$  and suppose you have say just  $a$  it is a real number purely a real number let say five  $\bar{z}$  is there is no change which is just five because it is lying on the real line

so its mirror images again on the line itself

so let us see some properties suppose say  $z = \bar{z}$  if and only if  $z$  is a real number or in other words

so the meaning of this one just the imaginary part of  $z$  is zero okay

so if we see that

so the  $z = \bar{z}$  which means after applying the mirror image we get the same number then it must lie on the real line that is what we are claiming

so if  $z$  is  $a + ib$  then suppose  $z = \bar{z}$  what does it mean it means that  $a + ib = a - ib$  then it is immediately clear that we are demanding two complex numbers to be equal then the component wise must be same

so the first components anyway it is equal

so  $ib = -ib$  just the by field axis you can cancel  $a$  then what you get is it is two times of  $ib = 0$  and this happens if and only if  $b = 0$  that is nothing but imaginary part of  $z$  imaginary part of  $z$  is zero ok

so what we observed is a simple proposition whenever  $z = \bar{z}$  then it must be a real number

so this particular say argument we will use frequently to say a complex number is a real number it is just enough to show that  $z = \bar{z}$  such idea will be frequently used to show a complex number is a real number property two if we apply conjugation for  $\bar{z}$  again

so we return back to  $z$  ok this is again it is a straight forward let  $z$  is  $a + ib$  then the  $\bar{z}$  is  $a - ib$  thus and this can be seen as just say minus  $b$  and  $z$  double bar

so again you take its conjugation

so what we see is that again say this is plus  $i$  and then minus of minus  $b$

so we see that it is again of course what  $i$  what we are doing is

so obvious one we are trying to write an argument third proposition if we consider sum of two complex numbers take its conjugation this is same as  $\bar{z_1 + z_2} = \bar{z_1} + \bar{z_2}$  this is again say easy to visualize say let us say that  $z_1$  is  $a + ib$  and  $z_2$  is  $c + id$  then  $\overline{z_1 + z_2}$  which is  $a + c + i(b + d)$  whole conjugation by definition this is  $a + c + i(b + d)$

minus  $i b$  plus  $d$  and which basically gives us which we can write it as  $a$  minus  $b$  plus  $c$  minus  $i d$  this is nothing but  $\bar{z}$  one bar plus  $z$  to bar fourth property that is it is always that for every  $z$  in complex number  $z$  into  $\bar{z}$  is always a real number is a non negative two real number

so it is clear that

so let us consider  $z$  as say general element which is  $a$  plus  $i b$  and we consider its conjugation by multiplication we can just verify that it is a square plus  $b$  square and each term is non negative

so the sum is again non negative

so from this particular proposition what we can observe suppose like a comment or a remark suppose you have  $z$  one  $z$  two their product is non zero ok then we can in fact conclude that both  $z$  one  $z$  two both are nonzero

so the proof i leave it as a exercise think about this

so what you observe the previous property when you multiply by its conjugation it gives a non negative real number you use this fact to conclude this particular remark fifth property that is  $z$  one into  $\bar{z}$  two let us  $\bar{z}$  one bar into  $z$  to bar it is easy to realize

so let me just consider  $z$  one  $\bar{z}$  two first you take the product and then take its conjugation which is same as take the conjugation for each complex number then do the product both will be the gives the same complex number

so let us consider first product of two complex numbers  $a$  plus  $i b$  multiplied with  $c$  plus  $i d$  and then take its conjugation we know what is this product this is  $ac$  minus  $b d$

so after conjugation you are going to get minus  $i a d$  plus  $b c$  you can verify that this product is nothing but coming from  $a$  minus  $i b$  and  $c$  minus  $i d$  which is nothing but the  $\bar{z}$  one bar into  $\bar{z}$  two bar the next proposition  $z$  inverse conjugation which is same as first take the conjugation and then take its inverse

so what we are asking is if you take the conjugation for  $z$  inverse what do you get it is basically same as first you take the conjugation for the complex number and then take its inverse

so its like this operation is commutative right let us just try to realize this

so what we have by definition  $z$  inverse it means that of course here we need  $z$  must be non zero

so by definition  $z$  inverse it means that multiplying with  $z$  we get one and now you apply conjugation by the above property

so we apply conjugation for this product element right hand side is a real number

so it gives the same element and the product conjugation is same as you take the conjugation first and then take its product its again gives one it means that  $\bar{z}$  inverse bar is inverse for the  $\bar{z}$  bar ok

so that is exactly what you like to prove this concludes that  $\bar{z}$  bar inverse is same as  $\bar{z}$  inverse bar right the seventh property which is to consider  $z$  one by  $z$  two take its conjugation this will be same as  $\bar{z}$  one bar divided by  $z$  to power again to make division make sense we need to assume that where  $z$  two is non zero

so the this relation again it follows from the above one

so combining the five and six say from five and six we can derive the following that is  $z$  one by  $z$  two the whole conjugation is given by thus can be realized as  $z$  one product with  $z$  two inverse

so i am writing the  $z$  two inverse is nothing but i am just this again a notation one by  $z$  two it means it is  $z^2$  inverse

so its bar proposition 5 says that this can be taken bar to each factor that is ok this is product with  $z^2$  inverse bar and this basically we get it as because the above proposition we see that it is  $\bar{z}$  one bar multiplied with  $z$  to bar the

inverse let me again just repeat that

so this is  $\bar{z}$  finally what we concluded is  $z^{-1}$  into  $z$  to  $\bar{z}$  the inverse as I mentioned this is nothing but the inverse factor we write it as  $1/z$   $\bar{z}$

so proposition eight maybe it is just a comment here or a remark when we write  $1/z$  ok the notation  $1/z$  it means that it is just the  $z^{-1}$  ok again what is  $z^{-1}$   $z^{-1}$  is an element which is when you take product with  $z$  it gives one okay

so in this particular sense we write it as  $z^{-1}$  as  $1/z$  ok just to cancel

so that you get one

so it is just a notation that  $1/z$  it means that it is  $z^{-1}$  and one more further remark here that

so once we commented about  $z^{-1}$  let us just see that  $1/z$  we can multiply and divide by  $\bar{z}$  then what you get is it is  $\bar{z}$  divided by  $z$  into  $\bar{z}$

so we know what is this factor this is  $a - ib$  and  $a^2 + b^2$  where  $z$  is written by  $a + ib$

so what we if you recall when we are when we calculated  $z^{-1}$  we try to solve the equation to find out the value for the  $z^{-1}$  now we see that by using the factor  $\bar{z}$  we are able to compute the  $z^{-1}$  easily just by multiplication and dividing by  $\bar{z}$  we are able to calculate  $z^{-1}$  that is  $\frac{a - ib}{a^2 + b^2}$  proposition eight this is simple one that is real part of  $z$  can be written as  $\frac{z + \bar{z}}{2}$  similarly the imaginary part of  $z$  can be written as  $\frac{z - \bar{z}}{2i}$  ok it is just merely the definition that if we consider  $z$  this is nothing but real part of  $z$  plus  $i$  times imaginary part of  $z$  and its conjugation is real part of  $z$  minus  $i$  times imaginary part of  $z$  now it is clear that real part of  $z$  is nothing but  $\frac{z + \bar{z}}{2}$  and similarly imaginary part of  $z$  is  $\frac{z - \bar{z}}{2i}$  let us do a simple problem complex number  $x + iy$  satisfies this equation that is square root of  $a + ib$  plus  $c + id$  then show that these numbers  $x$   $y$  that is  $x^2 + y^2$  the whole square gives you  $a^2 + b^2 + c^2 + d^2$

so what do you see is that suppose the complex number  $x + iy$  that is equal to square root of  $a + ib$  divided by  $c + id$  then we are able to derive this relation

so we need to prove this I say that  $z$  is a square root of some  $b$  by definition  $z^2$  is  $b$  ok

so we know what is the meaning of  $z^2$  and that should be equal to the  $b$  ok so whenever there is a  $z$  satisfies this equation then we write it as  $z = \sqrt{b}$

so by given assumption what we get is it is  $x + iy$  the whole square must be equal to  $a + ib$  divided by  $c + id$

so now the complex number

so if you see the identity there is no complex values involved just finally it is a real number equal to this factor

so now as you remember the one of the property is when we multiply  $z$  with  $\bar{z}$  that gives a non negative real number ok that gives a hint that maybe we can multiply with the factor which is exactly its conjugation

so the conjugation for the same complex number here let us say this is that we are multiplying by  $\bar{z}$  then again by the previous properties when you take  $z^{-1}$   $z$  to  $\bar{z}$  is given by  $z^{-1}$  multiplied by  $\bar{z}$

so by this relation you will see that immediately it is  $x^2 + y^2$  square multiplied by  $x^2 + y^2$  whole square and now this can be done by the

associative law

so you just write down what is the meaning of this this is  $x + iy$  multiplied by  $x + iy$  further here  $x - iy$  multiplied by  $x - iy$  we know the product it is  $x^2 - (iy)^2$  multiplied by its conjugation gives  $x^2 + y^2$

so we get  $x^2 + y^2$  the whole square right

so this is basically the left hand side

so the l h is what we considered and similarly correspondingly the right hand side what do you have  $x + iy$  the whole square is  $a^2 + b^2 + c^2 + d^2$  and we are taking its corresponding conjugation conjugation we know  $z^{-1}$  by  $z^{-2}$  bar is given by  $z^{-1}$  bar divided by  $z^{-2}$  bar which means it is  $a - ib - c - id$  and the product it gives  $a^2 + b^2 + c^2 + d^2$

so by given assumption these two are equal

so we get the required relation

so that is  $x^2 + y^2 = a^2 + b^2 + c^2 + d^2$

so using the property of multiplication one can derive the following simple identities namely  $(z + z^2)^2$  let me just write down directly what we have this product this is  $(z + z^2)$  product with  $(z + z^2)$  by distributive law  $z$  is multiplied with  $(z + z^2)$  plus again  $z^2$  multiplied with  $(z + z^2)$  again you use the distributive law

so we are using twice distributive law that is  $z^2 + z + z^3 + z^2$  plus  $z^2$  multiplied by  $(z + z^2)$  square and the product is commutative so these two factors are equal  $z^2 + z + z^3 + z^2$

so what we derived is like in the real line real numbers when we take  $a + b$  the whole square we get  $a^2 + 2ab + b^2$  same formula holds for the complex number as well and

so which means once you see this result then you can say prove the other identities like  $(z + z^2)^3 = z^3 + 3z^2 + 3z + z^3$  and  $(z - z^2)^2 = z^2 - 2z + z^2$  can be written as  $z^2 - 2z + z^2$

so what we are essentially trying to say is when you are doing this operation it is not necessary that you combine first real part you sum and then again the imaginary part you sum up and then you take the product this is not necessary

so this is basically like whatever identity it gives it is similar to the real number system we are able to work our pro say algebraic operations one simple exercise let  $z_1, z_2$  be two complex numbers then show that  $z_1$  multiplied by  $z_2$  bar and then combine  $z_1$  bar to thus is a real number now let us do a another problem here we are given with say three complex numbers if the sum of sum and product of three complex numbers are real then which of the following is possible following is r possible for some say possible for some pair of three tuple  $z_1, z_2$  in complex numbers

so what is given here we are considering three complex numbers their sum and product gives a real number

so now our question is which of the following is possible first choices exactly one of the three numbers is non real exactly two of the three numbers are non real third choice all three numbers are non real fourth choice all three numbers purely imaginary and non zero

so let us look back our question we are given with three complex numbers  $z_1, z_2, z_3$  their sum is real and their product is real number now let us look at the first choice whether this is possible exactly one of the three numbers is non real is it possible answer is no

so  $a$  is  $a$  is not possible

so here it here the statement says you consider at least oneness imaginary number remaining or can be real number

so which means that whenever we have a set with  $z_1$  which is not a real number that is  $z_1$  not equal to  $\bar{z}_1$  which means it is a complex number

so that is the imaginary part is not zero ok and the remaining say  $z_2$  can basically choose it as real numbers let me call it as  $a$  and  $b$  where  $a$  and  $b$  from real numbers then immediately i see that  $z_1$  plus  $z_2$  plus  $z_3$  what you are having is just the  $z_1$  plus  $a$  plus  $b$  certainly it is not a real number the reason is  $z_1$  contains a imaginary part which is not cancelled by other term

so that remains as it is

so which means it is not a real number

so the possibility of  $a$  is ruled out let us see the possibility of  $b$  exactly two of the three numbers are non real ok

so whether we could provide two set of pairs which is say they are complex numbers together a real number will satisfy our given condition ok

so let us see that

so what we are demanding is you give me three pair

so the sum must be real i am just repeating the what is our given assumption ok

so we are looking for a triple which satisfies this condition

so here what we can do is we can make a choice once you have a  $z_1$  we can take  $z_2$  as its conjugation then their sum become a real number

so  $z_3$  we can live as it is a real number then you have basically like it looks this is possible

so which means i take  $z_1$  as fixed example i take  $z_2$  is its conjugation and  $z_3$  is let us say just simply one then what i know is their sum is real and together  $z_3$  gives a real number further when you are taking the product  $z$  into  $\bar{z}$  we know it is a non negative real number then product with  $z_3$  again a real number

so this is basically like it satisfies yes this is possible

so choice  $c$  all three numbers are non real it is again possible yes

so just basically like i give you one set of pair maybe you can try with the other set of pair where it satisfies this relation

so the one set is just consider one minus  $i$   $z_2$  as again the same number and  $z_3$  now you see i am just going to manipulate

so when i sum it up what i get is say here minus  $2i$

so what i need to do is when i sum it up it must be a real number

so naturally i make a choice  $2y$  but further of course we need to see that whether the product is a real number when you do the product

so  $z_1$  into  $z_2$  we see that it is say exactly  $z^2$

so  $z^2$  it is say one square plus  $i$  square that is minus one and minus two times of  $i$

so what you get is minus two  $i$  now you when you product say what you getting is a real number

so what is the exercise for you is you find the other set of pair where it satisfies this  $d$  is again i leave it as a exercise but i will write down the answer answer is again this choice is not possible but you try to do it

so let me just summarize what we have done is we introduced conjugation of a complex number and we studied several properties now what we will discuss is modulus of a complex number

so first let me recall how do we define the modulus for real number system

so for  $a$  in real numbers we define modulus of  $a$  as  $|a|$  if  $a$  is non negative minus  $a$  if  $a$  is less than zero ok the same one we could also write it as maximum of  $a$  together with minus  $a$  ok

so this gives the modulus definition now question is similarly can we do for the complex number system

so once we see the first definition or even the second definition they related with the relation that is basically you are comparing a number with  $0$  then we basically define this modulus question is can we really have a say the less than relation in the complex number system answer is no

so let me just have a quick say idea that we cannot have a sort of a relation that is the ordering what we have in the real number which cannot be possible in the complex number system

so i am not writing in detail but let me just give a some rough idea ok  
so idea for

so first let me write there is no less than which means we cannot compare two complex numbers in  $\mathbb{C}$  ok

so just ah idea is if suppose there is a relation less than ok then if suppose there is a relation in  $\mathbb{C}$  then what will happen any number can be compared with the other number for example in this case

so either

so what we end up immediately either is less than zero or  $i$  is greater than zero

so once we have this of course we no need to basically like consider this particular relation directly we could start with say either this or actually we could directly say that  $i^2$  must be greater than zero the reason is whenever you have a order relation we can show that a number multiplied by itself will be non-negative if there is a order relation okay

so which means  $i^2$  must be greater than  $0$  but  $i^2$  we know it is minus 1 which is less than  $0$

so which means this particular relation will tell you that minus 1 is greater than  $0$  but we have this that is contradiction okay

so what we are encountering we are encountering that we are not able to define the modulus as usual like in the real line but we can basically try to see what it physically represents then we may be can associate modulus in a different sense

so when we are taking a real line there is a  $a$  it may be a positive number maybe a negative number the mod  $a$  always say mentions the distance from zero ok

so similarly like if it is say whatever you have if you have a negative number let us say minus  $b$  then still basically like what it says is mod  $b$  says the distance between zero and minus  $b$  ok

so with this distance as an idea we can try to think for a modulus for a complex number

so let  $z$  be  $a + i b$  then consider point  $z$  equal to  $a + i b$

so here my say the length here it is  $b$  which is basically the unit here which is  $i b$  and here the length is  $a$  and what we have zero now we are trying to associate mod  $z$  ok

so we would like to define modulus of  $z$  that is denoted by this notation by pythagoras theorem we know that what is the distance for this this is distance is given by square root of  $a^2 + b^2$

so it makes sense now we are trying to define modulus as a distance from the origin to the point

so the modulus which is defined as square root of  $a^2 + b^2$  now what we observe is immediately that it is always a non negative number and one more point

so if we see the association to the  $r$  two plane then this point we know that it is  $a + i b$  and this point we know that it is zero comma zero then the distance between these two points which is given by euclidean distance which is also coincide with the what we have written

so what we are doing is it is consistent with the what we have the association with the  $r$  two plane

so let us see simple example that suppose  $z$  is a one plus  $i$  then  $\text{mod } z$  by definition it is  $1^2 + 1^2$

so you get  $\sqrt{2}$  and if suppose  $z$  does just a real number let us say  $5$  then  $\text{mod } 0$  is just the  $\phi^2$  again what we associate is the positive number because we see we are seeing it as a distance

so which is  $\phi$  and if  $z$  is some purely  $i$  ok let us just consider  $z$  equal to  $i$   $\text{mod } z$  is square root of one square right

so with this simple examples let us see some properties for modulus function let me just summarize what we have done we introduce conjugation of a complex number and we studied its properties now we introduce the modulus of a complex number which associates a distance from the origin to the complex number the further properties we will discuss in the next lecture thank you