welcome to the iit problem solving session on quadratic equations in the series of coming three lectures we are going to solve some problems on quadratic equations our questions will be mainly of mcq types and sometimes we will see more than one options are correct also we will see at times a pictorial illustration helps us to solve a problem much easily we shall start by reviewing the theory of quadratic equations briefly by quadratic equation we mean a polynomial equation of degree 2 and the word quadratic came from the latin word quadratus which means made square

so let us first write a general form of a quadratic equation a x square plus b x plus c is equal to 0 where a b and c are complex numbers with a being non-zero because note that if a is equal to 0 then the polynomial equation will no more be of degree 2.

so we must have a to be non-zero now if a is real then without loss of generality we can take a to be positive because if a is negative then we know minus of a is positive

so we can multiply the polynomial equation by minus 1 and we would get the coefficient of the term x square in the polynomial equation is positive this quadratic equation is having exactly two solutions namely minus b plus minus square root of b square minus divided by 2a and i will give a brief hint how to derive that this equation is having these two solutions let us call minus b minus square root of b square minus 4ac by 2a as alpha and minus b plus square root of b square minus 4ac by 2a as beta

so our equation is ax square plus bx plus c is equal to 0 if we take a out we get x square plus b by a we can divide by a because a is non-zero into x plus c by a is equal to 0

so therefore this is same as a into x square plus 2 into x into b by 2 a plus b square by 4 a square minus b square by 4 square plus c by a is equal to 0

so we have a into x plus b by 2 a whole square minus square root of b square minus 4ac divided by 2a whole square is equal to 0.

so now if we use the formula of x square minus y squared and use the fact that a is non-zero we get that alpha and beta are the solutions of this quadratic equation ax square plus bx plus c is equal to 0 now this quantity b square minus 4 ac is generally denoted by d and this has a name it is generally called the discriminant of the polynomial a x square plus bx plus c

so if we write using the notation d then we have alpha is equal to minus b minus square root of d divided by 2a and beta is equal to minus b plus square root of d divided by 2a let us make a small note here if a b and c are all rational numbers then we can say if t is a square then alpha beta are rational numbers and secondly if d is not a square then alpha beta are conjugate swords okay now let us assume a b and c all of them are real numbers and our standing assumption is a is always positive

so let us note a few facts here if the discriminant d is strictly bigger than 0 that is if d is positive then alpha and beta both the solutions are distinct and they are real secondly if d is 0 then we get alpha is equal to beta and also they are real and the last thing if t is strictly less than 0 that is if d is negative then we get alpha and beta both are distinct and they are complex conjugates of each other let us keep these facts in mind because these facts will be useful for our problem solving sessions as i said in the beginning sometimes visualizing something pictorially makes our life simpler

so now we are going to draw the graph of the parabola y is equal to a x square plus b x plus c where a b and c all of them are real numbers with a being strictly positive and we will have three cases here let us take our first case to b the discriminant d is strictly bigger than zero and in every case we will have a number of sub cases here let us take our first sub case to be b is strictly less than zero and c is strictly bigger than zero now i draw the x axis this is our y-axis and the parabola is like this this is the vertex let us call it by p and this is having coordinates minus b by 2 a comma minus d by 4 a and this is y intercept 0 comma c this is alpha and this is beta

so here we can see that alpha and beta both are real and they are distinct our next sub cases b is strictly less than zero and c is strictly less than zero

so this is the x axis and this is our y-axis and we draw the parabola here is the vertex p this is y intercept this is alpha this is beta

so here also alpha and beta both are real and distinct now our third sub case is c is strictly less than zero and b is strictly bigger than 0

so this is x axis this is y axis and this is the parabola

so this is the point p which is vertex this is y intercept this is alpha and this is beta

so alpha and beta both are real and distinct now i will do one more sub case and leave the rest for you to try our fourth sub case is c is strictly bigger than zero and b is strictly bigger than zero now this is x-axis this is y-axis and the parabola is like this

so here is our vertex this is y intercept this is alpha and this is beta

so in this subcase also we see that alpha and beta both are real and distinct now you shall try this the corresponding sub cases for b is equal to 0 or c is equal to 0

so in the case 1 which is the discriminant d is strictly bigger than 0 we have the solutions alpha and beta both are real and distinct now our second case is the discriminant d is equal to 0 here we take our first sub case to be c is strictly bigger than 0 and b is strictly less than 0

so here is the x axis here is the y-axis and the parabola is like this

so this is y-intercept 0 comma c this is vertex b and alpha and beta are here the coordinates of the vertex p are alpha comma 0 which is same as beta comma 0

so in this sub case alpha and beta both are real and they are same in fact we can more clearly write p is having coordinates minus b by 2 a comma 0 and here alpha is equal to beta is equal to minus b by 2a second sub cases c is strictly bigger than 0 and b is strictly bigger than 0 here the parabola is like this this is y intercept 0 comma c here is the vertex p and here is also alpha is equal to theta now our third sub case is c is equal to 0 and b is equal to 0 this is the last sub case in this case because d is equal to b square minus 4ac is equal to 0 that means b square is equal to 4ac and since b square is always bigger than equal to 0 we have ac is bigger than equal to 0 and as a is always strictly bigger than 0 we get that c is bigger than equal to 0 now this is our x-axis and this is y-axis and here the parabola is like this

so this is the vertex p which is having coordinate 0 comma 0 and here

so alpha is equal to beta is equal to 0

so in this case we see that the solutions are always real and they are equal now we move on to the last case which is the discriminant d is strictly less than zero we take our first sub case here to be b is strictly less than zero and c is strictly bigger than zero

so this is our vertex p this is y intercept and from this picture we can clearly see that there is no real solution as the parabola is not intersecting the x axis

so here alpha beta are not real okay second sub case is b is strictly bigger than zero and c is strictly bigger than zero and here the parabola looks like this

so the vertex p is here and y intercept is here and here also no real solution the third sub case is b is equal to 0 and c is strictly bigger than 0 now let us draw the parabola

so this is x-axis and this is y-axis and here the parabola is like this

so here is the vertex which is also y-intercept

so this is 0 comma c and here also no real solution exists also let us now note that no more sub cases are left because here d is equal to b squared minus 4 ac is strictly less than 0

so therefore b square is strictly less than 4ac

so from here we are getting that c is strictly bigger than b square divided by 4 a now we know that b square is always bigger than or equal to 0 and a is strictly bigger than 0

so from these two information we can conclude that b square divided by 4 a is bigger than or equal to 0 and therefore we have c is bigger than or equal to zero therefore we have completed seeing all the cases pictorially one also studies sign of the solutions alpha and beta when a b and c all of them are real numbers recall that for d strictly less than 0 we have already seen there is no real solution

so for d strictly less than 0 we cannot study the sign of alpha and beta we assume d to be bigger than or equal to zero let us consider our first case as d is equal to zero in this case we have alpha is equal to beta is equal to minus b divided by 2a

so if b is strictly bigger than 0 then in this case we have alpha and beta both of them are strictly less than 0 if b is strictly less than 0 then we get alpha and beta both of them are strictly bigger than 0 and if b is equal to 0 then we get alpha is equal to beta is equal to 0 now we consider d to be strictly bigger than 0 we know the solutions are minus b plus minus square root of d divided by 2a now as a is positive the quantity 2a is strictly bigger than 0

so for us it is enough to study the sign of the quantity minus b plus minus square root of d to study the sign of the solutions alpha and beta let b to be strictly bigger than 0

so here one of the solutions in particular which we called as alpha minus b minus square root of b square minus 4ac divided by 2a is always strictly less than 0 that is alpha is always negative

so let us look at what happens to the other solution beta now if c is also strictly bigger than 0 then d which is b square minus 4 ac is strictly less than b square

so therefore we get that square root of d is strictly less than b as d and b both are strictly bigger than 0

so therefore from here we can write that minus b plus square root of d is negative

so therefore beta which is minus b plus square root of d divided by 2a is negative if c is equal to 0 then we have beta is equal to 0 because beta is minus b plus square root of b square minus 4 ac

so we have only square root of b square here divided by 2a now we have b is strictly bigger than 0

so we only get minus b plus b divided by 2a which is 0 if c is negative that is c is strictly less than 0 then we have d is equal to b square minus 4 ac is strictly bigger than b square

so therefore square root of d is strictly bigger than b as d and b both are positive therefore we have now beta is equal to minus b plus square root of d divided by 2a is strictly bigger than zero that is beta is positive now if b is strictly less than zero then we have the solution beta which is minus b plus square root of d divided by 2a is strictly bigger than zero that is the sign of the solution beta is positive so we have to look at the sign of the solution alpha now late c to be strictly bigger than 0 then here we have d is equal to b square minus 4 ac is strictly less than b square

so therefore we can say that square root of d is strictly less than minus b as d is strictly bigger than 0 and b is strictly less than 0 and from here we can conclude that minus b minus square root of d is strictly bigger than 0

so therefore alpha which is minus b minus square root of d divided by 2a is strictly bigger than 0 that is the sign of the solution alpha is positive if c is equal to 0 then we can see that alpha is equal to 0 because alpha is nothing but minus b minus square root of b square divided by 2a and as b is strictly less than 0 we get here minus b plus b divided by 2a which is equal to 0.

now if c is strictly less than 0 that is if c is negative then d is strictly bigger than b square

so therefore we have square root of d is strictly bigger than minus b and hence we can say that alpha is strictly less than zero now if b is equal to zero then we know the solutions are plus minus square root of d divided by 2a and as we have here d is strictly bigger than 0 we can clearly see alpha which is minus square root of d divided by 2a is strictly less than 0.

and beta which is square root of d divided by 2a is strictly bigger than 0

so the sine of alpha is negative and the sine of beta is positive now i ain't recalling the theory part by noting down a few equalities

so we have a quadratic equation ax square plus bx plus c is equal to 0 where a b and c are all complex numbers we have sum of the solutions alpha and beta is equal to minus b divided by a if we sum alpha and beta we get this is nothing but minus b by a the second one is that the product of the solutions alpha and beta is b divided by 2 a whole square minus square root of b square minus 4 ac divided by 2a whole square

so this is equal to c divided by a and the third thing is that the distance between the solutions alpha and beta is modulus of square root of b square minus 4 ac divided by a and the last equality a alpha square plus b alpha plus c is equal to 0 and a beta square plus b beta plus c is equal to 0 now with all these we start solving some problems on quadratic equations in this question we are given that theta is an angle which is strictly between minus pi by 6 to minus pi by 12

so from here we know that theta lies in the fourth quadrant we are told that alpha one and beta one are the solutions of the quadratic equation x square minus two x sig theta plus one equals to 0 and alpha 2 beta 2 are the solutions of the quadratic equation x square plus 2 x tan theta minus 1 equals to 0 we are also told that alpha 1 is strictly bigger than beta 1 and alpha 2 is strictly bigger than beta2 then our job is to find out what is alpha 1 plus beta2 and to do that we have to find out what are alpha 1 beta 1 and alpha 2 beta 2

so we first consider the quadratic equation x square minus 2 x sig theta plus 1 equals to 0 and the solutions of this quadratic equation are 2 sec theta plus minus square root of 4 6 squared theta minus 4 divided by 2 and this is equal to sig theta plus minus tan theta now since theta is in the fourth quadrant we know that tan theta is negative therefore we can say that sec theta minus tan theta is strictly bigger than sec theta plus tan theta

so we now know that alpha one is sec theta minus tan theta and beta 1 is sec theta plus tan theta next we consider the second quadratic equation which is x square plus 2 x tan theta minus 1 equals to 0 and the solutions of these equations are minus 2 tan theta plus minus square root of 4 tan square theta plus 4 divided by 2 and this is equal to minus tan theta plus minus sec theta again since theta is in the fourth quadrant we know that sec theta is positive

so therefore we can conclude that minus tan theta plus sec theta is strictly

bigger than minus tan theta minus sig theta
so now we also know that minus tan theta plus sig theta is alpha 2 and minus tan theta minus sec theta is beta 2
so now with all this we are ready to find out what is alpha 1 plus beta 2 we write what is alpha 1 plus beta 2 this is sig theta minus tan theta minus tan theta minus sig theta which is minus 2 tan theta
so therefore we have now alpha 1 plus beta 2 is equal to minus 2 tan theta now we go back to the question in the question we see that the third option is the correct one now we look at this question we have the set ace which consists of all the non-negative real numbers x
so that x satisfies the equation 2 into modulus of square root of x minus 3 plus square root of x into square root of x minus 6 plus 6 equals to 0 we have to find out how many elements are there in the same s and to do that we shall first note that square root of x minus 3 is bigger than equals to 0 when x is bigger than equals to 9 and square root of x minus 3 is strictly less than 0 when x is strictly less than 9 now we consider the case x bigger than equals to 9 first in this case we rewrite the equation
so our equation is 2 into square root of x minus 3 plus square root of x whole square minus 6 square root of x plus 6 equals to 0
so therefore we have the equation square root of x whole square minus 4 square root of x is equal to 0
so this is a quadratic equation in the variable square root of x as the constant term is 0 here we can solve it very easily
so this is square root of x into square root of x minus 4 equals to 0
so from here we can conclude that square root of x is equal to 0.
or square root of x is equal to 4 and from here we get that x is equal to 0 or x is equal to 16.
now recall that we are in the case x bigger than equals to 9
so therefore x equals to 16 is possible but x equals to 0 is not possible in this case now we consider the next case that is the case when x is strictly less than 9 and in this case our equation becomes minus 2 into square root of x minus 3 plus square root of x whole square minus 6 square root of x plus 6 is equal to 0.

so therefore we have square root of x whole square minus 8 square root of x plus 12 is equal to 0
so again we have a quadratic equation in the variable square root of x we solve it for the variable square root of x and we get the solutions are 8 plus minus square root of 64 minus 48 divided by 2
so we have square root of x is equal to 2 or 6.

so from here we get that x is equal to 4 or x is equal to 36 now since x is strictly less than 9 in this case
so x equals to 36 is not possible and x equals to 4 is possible
so from here we get only two possible choices of x that is x equals to 16 and x equals to 4
so therefore we can say that the set is is containing exactly two elements
so therefore the option two is the correct one here is the third question and in this question we are given the equation modulus of x square plus 4x plus 3 plus 2x plus 5 is equal to 0 and we have to find out how many real solutions does this equation have
so first we solve this equation for all those values of x for which x square plus 4x plus 3 is bigger than equals to 0 and in this case our equation becomes x square plus 6 x plus a is equal to 0 now when we solve this quadratic equation

for x we obtain that x is equal to minus 6 plus minus square root of 36 minus 32 divided by 2 which is same as minus 6 plus minus 2 divided by 2

so we have x is equal to minus 2 or minus 4 now we shall check that among these two values of x for which the condition x square plus 4 x plus 3 bigger than equals to 0 is satisfied we rewrite the condition here again x square plus 4 x plus 3 is bigger than equals to 0 first we substitute the value x is equal to minus 2 in the quadratic polynomial x square plus 4 x plus 3 and we get that 4 minus 8 plus 3 this is equal to minus 1 which is strictly less than 0 therefore for this value of x that is x equals to minus 2 the condition is not satisfied therefore x equals to minus 2 is not possible next we take the value x equals to minus 4 and when we substitute this value in the quadratic polynomial we get that 16 minus 16 plus 3 and this is equal to 3 and clearly this is bigger than equals to 0

so therefore x equals to minus 4 is possible now we solve the equation for all those values of x for which x square plus 4 x plus 3 is strictly less than 0 and in this case our equation becomes minus x square minus 4x minus 3 plus 2 x plus 5 v is equal to 0 that is we have x square plus 2 x minus 2 is equal to 0

so we solve this quadratic equation now and we obtain that x is equal to minus 2 plus minus square root of 4 plus 8 divided by 2 and this is same as minus 2 plus minus 2 square root of 3 divided by 2.

so we have x is equal to minus 1 plus minus square root of 3 now among these two values of x minus 1 plus minus square root of 3 we shall check for which the condition x square plus 4 x plus 3 strictly less than 0 is satisfied first we consider x is equal to square root of 3 minus 1

so when we substitute this in the quadratic polynomial x square plus 4 x plus 3 we obtain that 3 minus 2 square root of 3 plus 1 plus 4 square root of 3 minus 4 plus 3 and this is same as 2 square root of 3 plus 3 which is strictly bigger than 0

so therefore x is equal to square root of 3 minus 1 is not a solution then we consider x is equal to minus 1 minus square root of 3

so when we substitute these we obtain that square root of 3 plus 1 whole square minus 4 into square root of 3 plus 1 plus 3 this is same as 3 plus 2 into square root of 3 plus 1 minus 4 into square root of 3 minus 4 plus 3 and this is same as 3 minus 2 into square root of 3 and clearly this is strictly less than 0 as 12 is strictly bigger than 9 therefore x is equal to minus 1 minus square root of 3 is possible

so we get that this equation is exactly having two real solutions

so the third option is correct here in this question we are given the equation e to the power sine x minus e to the power minus sine x minus 4 is equal to 0 and as it is evident from the options we have to find out how many real solutions does this equation have

so we write the equation e to the power sine x minus e to the power minus sine x minus 4 is equal to 0 and this is same as writing e to the power 2 sine x minus 4 into e to the power sine x minus 1 is equal to 0

so we see that this is a quadratic equation in e to the power sine x for simplicity we put y is equal to e to the power sine x

so we have this quadratic equation y square minus 4 y minus 1 is equal to 0

so when we solve it for y we obtain that y is equal to 4 plus minus square root of 16 plus 4 divided by 2 which is same as 4 plus minus 2 square root of 5 divided by 2 and this is same as 2 plus minus square root of 5 first we show that y cannot be equal to 2 minus square root of 5

so for that first we note that 2 minus square root of 5 is strictly less than 0 as 4 is strictly less than 5 now for any real number x sine of x is always real and as we know for any real number x e to the power x is always real and in fact it is strictly bigger than 0

so therefore e to the power sine x is strictly bigger than 0 for any real number x

so here as y is equal to e to the power sine x

so y cannot be equal to 2 minus square root of 5 as 2 minus square root of 5 is strictly less than 0.

so the only possible choice for y is y is equal to 2 plus square root of 5 now we write what is y this is e to the power sine x

so we have e to the power sine x is 2 plus square root of 5 we take logarithm to the base e on both sides of this equation and therefore we obtain sine x is equal to logarithm of 2 plus square root of 5 now 2 plus square root of 5 is strictly bigger than 4 and we know 4 is strictly bigger than e

so therefore logarithm of 2 plus square root of 5 is bigger than logarithm of e as logarithm is an increasing function and we know that logarithm of e to the base e is equal to 1

so therefore we are getting that sine x is strictly bigger than 1 which is not possible as we know that for any real number x sine x is always bigger than equals to minus 1 and less than equals to plus 1

so therefore sine is bigger than 1 is not possible

so we get that this equation is not having any real solutions this is our fifth question in this question we are given three positive real numbers p q and r which are in an arithmetic progression also we have this quadratic equation p x square plus q x plus r is equal to zero we have to find out the conditions on p and r for this quadratic equation to have all real solutions first note that q is equal to p plus r divided by 2 because we are told that p q and r are in an arithmetic progression now as we know the necessary and sufficient condition for the quadratic equation p x square plus q x plus r is equal to 0 to have all real solutions is q square minus 4 p r is bigger than equal to 0

so when we substitute the value of q which is p plus r divided by 2 in this condition we obtain p plus r whole square minus 16 p r is bigger than equal to 0 which is p square minus 14 p r plus r square is bigger than equal to zero now since p is a positive real number we know that p is non-zero

so therefore we can divide this inequality by p square and we obtain 1 minus 14 r by p plus r by p whole square is bigger than equal to zero we now rewrite this inequality r by p whole square minus 2 into r by p into 7 plus 49 is bigger than equal to 48

so therefore we have r by p minus 7 whole square is bigger than equal to 48

so therefore we can conclude that modulus of r by p minus 7 is bigger than equal to 4 square root of 3

so we can see that the fourth option modulus of r by p minus 7 is bigger than equal to 4 square root of 3 is correct now let us take the other three options to do that let us first find out the location of the ratio r by p

so we can write from this inequality that r by p is bigger than equal to 7 plus 4 square root of 3 and also we can write minus r by p plus 7 is bigger than equal to 4 square root of 3 which means r by p is less than equal to 7 minus 4 square root of 3

so from here we can now locate where does the ratio r by p belong to since r and p both are positive numbers the ratio r by p is positive

so r by p belongs to open 0 to closed 7 minus 4 square root of 3 union closed 7 plus 4 square root 3 to infinity now observe that when the ratio r by p is in the interval open 0 to closed 7 minus 4 square root of 3 then the ratio p by r is in the interval closed 7 plus 4 square root of 3 to infinity and when the ratio r by p is enclosed 7 plus 4 square root of 3 to infinity then the ratio p by r is in the interval open 0 to closed 7 minus 4 square root of 3

so from here now we can see that the condition modulus of r by p minus 7 is bigger than equal to 4 square root of 3 is equivalent to modulus of p by r minus 7 is bigger than equal to 4 square root of 3

so the option 3 is also correct now in the process as we have seen that we are getting possible choices for the ratios p by r and r by p in certain intervals

so the option 1 and option 2 are not correct

so this solves the question five here in this question we are given three positive real numbers a b and c which are the lengths of the sides of a triangle and also we are given one real number lambda and the quadratic equation x squared plus 2 into a plus b plus c into x plus 3 lambda into a b plus bc plus ca is equal to 0 we are to find out the range of lambda for which this given quadratic equation is having all real solutions now to do that we shall first write down the necessary and sufficient condition for this quadratic equation to have all real solutions and the condition is 4 into a plus b plus c whole square minus 12 lambda into a b plus bc plus ca bigger than equals to zero now if we cancel four from this inequality and if we split this term a plus b plus c whole square we obtain that a square plus b square plus c square plus 2 into a b plus b c plus c a minus 3 lambda into a b plus b c plus c a is bigger than equal to 0

so therefore finally we obtain 3 lambda minus 2 is less than equal to a squared plus b squared plus c squared divided by a b plus b c plus c a now we use the information that a b and c are the lengths of the sides of a triangle

so therefore a plus b is strictly bigger than c and a plus c is strictly bigger than b and b plus c is strictly bigger than a

so we have these three inequalities now from the first inequality a plus b is strictly bigger than c we can write that a is strictly bigger than c minus b and also b is strictly bigger than c minus a from the second inequality we can write that a is strictly bigger than b minus c and c is strictly bigger than b minus a and from the last one we can write that b is strictly bigger than a minus c and c is strictly bigger than a minus b now let us consider this inequality that a is strictly bigger than c minus b and the inequality a is strictly bigger than b minus c

so from these two we obtain that a is strictly bigger than modulus of c minus b next we consider this inequality that b is strictly bigger than c minus a and the inequality that b is strictly bigger than a minus c

so from here we obtain that b is strictly bigger than modulus of c minus a and from the last two remaining inequalities we obtain that c is strictly bigger than modulus of b minus a now we are going to use all these three inequalities and what we will do is that we will square both sides of this inequalities then we get a square is strictly bigger than b square plus c square minus 2 b c and b square is strictly bigger than a square plus c square minus 2 a c and c square is strictly bigger than a square plus b square minus 2 a b now we add all these three inequalities then we obtain a square plus b square plus c square is strictly bigger than 2 into a square plus b square plus c square minus 2 into a b plus bc plus ca

so therefore we have a square plus b square plus c square divided by a b plus bc plus c is strictly less than 2 now recall that we already had that square plus b square plus c square divided by a b plus bc plus ca is bigger than equals to 3 lambda minus 2

so therefore we can say that 3 lambda minus 2 is strictly less than 2

so therefore lambda is strictly less than 4 divided by 3 hence the first option that lambda is strictly less than 4 by 3 is correct

so therefore the second option that lambda is strictly bigger than 5 by 3 cannot be correct and also at the same time we can say that the fourth option that lambda is in the open interval 4 by 3 to 5 by 3 is also not correct

so we have to only check whether the third option that is lambda is in the open interval 1 by 3 to 5 by 3 is correct or not to check that let us put lambda is equal to 0 in the given quadratic equation and putting this we obtain that x square plus 2 into a plus b plus c into x is equal to 0 and clearly we can see that this quadratic equation is having all real solutions as a b c are real numbers

so therefore the third option is also not correct as we see for 0 that is for lambda is equal to 0 this equation is also having all real solutions but 0 is not in this open interval 1 by 3 to 5 by 3

so here only one option is correct which is the first option that is lambda is strictly less than 4 by 3.

we end our first session here today we have reviewed the theory of quadratic equations and using that we have solved some problems in the next two lectures we will solve some more problems you