welcome to the third lecture on trigonometric functions in the last lecture we ended up with deriving an expression for cos of x minus y and cos of x plus y in terms of cos x cos y sine x and sine y

so we will continue with that and in this lecture we are going to derive expressions for sine two x sin three x cos two x cos three x in terms of sine x cos x and some other derivations also

so towards the end in the last lecture we proved that cos of x minus y is equal to cos x cos phi plus sign x sine y and then by substituting ah replacing this y here with minus y we got the other expression that cos x plus y is equal to cos x cos phi minus sin x sin y

so this can be used these two formulas can be used to find the cosine of some other angle apart from the well known angles for example we already know the cosine and sine for 45 degree and 30 degrees

so that can be used to find the cosine of 15 degrees by using this formula and having x equal to 45 degree and y equal to 30 degrees

so that is written here

so cos 45 minus 30 and then of course i use this formula here

so what we get is cos x which is cos 45 cos y plus sine 45 here sine 30.

so these are all degrees ok and this of course is equal to cos 45 is equal to 1 by root 2 and cos 30 is equal to root of 3 over 2.

sine 45 is also 1 over root 2 and sine 30 is half

so you get

so that is how you get cos of fifteen degrees is equal to one plus square root of three over two times root two now using these two basic formulas for cos x minus y and plus x plus y other other simplifications can also be derived for example here if we are given an expression of this type 2 cos x cos y then it can be written as

so let us focus on these two equations here

so if we add up these two equations on the left hand side we are going to get cos x minus y plus cos x plus and on the right hand side sin x sin y is going to cancel because as a plus here there is a minus here and then these two will get added up and we get two cos x cos y and therefore two cos x cos y is equal to cos of x minus y plus cos of x plus y and similarly if we compute cos x minus y minus cos x plus y then this term in both these equations will get cancelled and we will get two sin x sin y

so two sin x sin y is equal to cos x minus y minus cos x plus y

so later on when we do a lot of examples you will see that this kind of this type of formulas will be very handy and it will be good if you can memorize it

so we saw in the previous slide as to how we can compute cos of 15 degrees somebody might think that now that we have computed cos 15 degrees can we compute cos of half of 15 degrees which is cos of 7.

5 degrees and yes it is possible

so for to do that let us first derive an expression for cosine of two x in terms of cosine of x and then what we will do is we will put x equal to fifteen degrees half of fifteen degrees and then we will see that we can solve and find out cos of seven point five degrees

so now cos of two x can be written as cos of x plus x and then we use the formula on the previous slide for cos of x plus y but with y equal to x

so when you put y equal to x in this equation you get cos x into cos x minus sin x into sin x

so that becomes cos square x minus sin square x but we know that sin square x plus cos square x is equal to one for all x and therefore in this equation we can replace sine square x by one minus cos square x and that is the expression

that we get

so we now see that cos of two x is equal to two times square of cosine of x minus one

so if you happen to know the value of cosine of x you can compute the value of cosine of two x and in fact you can show that this is also equal to one minus two times sin square x to get that is very simple just you just need to go two steps back and in this equation over here you need to replace this cos square x term by 1 minus sin square x and that is how you get this term here this expression now let us see how we can calculate cos seven point five degrees given that we know the value of cos 15 degrees from the previous slide before that this equation can also be written in another way

so we can write two cos square x is equal to one plus cos two x and therefore cos x is equal to the ambiguity about plus and minus will depend on the value of x now let us try and compute the value of cos of x for x equal to seven point five

so we put x equal to seven point five in this equation here but we know that cos of seven point five degrees is a positive number or positive real number and therefore we are going to only take the positive square root

so this is the value of cos of 7.
5 degrees

so is very simple on the right hand side we just need to take the value of cos 15 degrees that we had calculated in the previous slide and substitute it here and then compute the square root of this expression using the cos x minus y formula many other simplifications can be done for example we will try to see what cos of x minus pi by 2 is

so this is cos x cos y

so here your y is equal to pi by two ninety degrees plus sin x sine y we know that cos of pi by two is zero and sine of pi by two is equal to one and therefore the first term here in this summation will be zero and only the second term is going to contribute

so this is equal to sine of x similarly cos of x plus pi by two is

so we will use now the formula for cos of x plus y with y equal to pi by two which is cos x cos y minus sin x sin y again here because cos of pi by two is zero this first term is equal to zero and

so what we get is minus sin x similarly we can compute expressions for sine of y plus pi by two and sine of y minus pi by two but you would be probably wondering that we haven't covered the expressions for sine of x plus y and sin of x minus y but with that is something that we will be doing in the next few slides but here even without that we should be able to compute sine of y plus pi by two if you go back to this equation here where we had written that cos of x minus pi by two is sine of x now if we replace this x over here

so we say that x is actually equal to y plus pi by 2 then what we get is sine of y plus pi by 2 which is this is equal to this expression but with x equal to y plus pi by two

so we are using this substitution here therefore sine of y plus pi by two is equal to cos of instead of x we put y plus pi by two and then minus pi by two here

so it is simply cos of y similarly we can find out what sign of y minus pi by 2 is we go back to this equation that we had just derived now cos of x plus pi by 2 is equal to minus sin x

so here what we do is we try to do the substitution x equal to y minus pi by 2

so what we get then is that minus of sine y minus pi by 2 is equal to cos now instead of x we have to put y minus pi by 2.

so we get there is a plus pi by two here but there is a minus sign here and since we do not have a minus sign here we need to put a minus sign over here
 so this becomes equal to minus cosine of y y
 so these four identities are very useful and we will see that when we do some example problems later on in this slide we are now going to derive similar to this expression for cos of x minus y equal to cos x cos y plus sin x sin y i am going to derive an expression for sine of x minus y in terms of cos x cos y sine x and sine y and for that we will use the result on the previous page where we had shown that sine of for any angle z sine of z equals if you
 so sine of x equals sine of cosine of x minus pi by 2.

 so sine of z is cosine of z minus pi by 2 cosine of z minus pi by 2.
 and here we will treat this as equal to z
 so by using this identity what we get is that sine of z is cos of z minus pi by 2 but z is x minus y
 so this is cosine of z minus pi by two which i can write as cosine of x minus y plus pi by 2
 so now if you see this expression here is of the form cos of a minus b or cos of x minus y where i will be treating this as the new y and then we use this formula
 so we get this to be equal to cos x cos nu y
 so this is the new y
 so cos x cos y cos x cos nu i and then plus sign x times sine of nu y now from the previous slide we know that cos of y plus two cos of y plus pi by two cos of y plus pi by two is minus sin y
 so i am just going to replace this x over here with y
 so this thing this term here becomes cos x and then there is a minus sign y plus sine x and then again from the previous slide we had derived that sine of y plus pi by two is the same as cos of y
 so using that we have sine of y plus pi by 2 and that is the final expression which is that sine of x minus y is equal to sine x cos y minus cos x sin y using this formula we can now easily derive the expression for sine of x plus y we just need to replace this y with minus y for sine x plus y we write this as sine of x minus minus y
 so this minus y is like the the y that we had in the expression for sine of x minus y from the previous slide we know that sine of x minus y is sine x cos y minus cos x sine y and therefore here just replacing this y with minus y will get sine of x plus y equal to sine x cos of minus y minus cos x sine of minus y now we know that cosine is an even function and sine is an odd function therefore cosine of minus y is cos y and sine of minus y is minus sine y using these two facts we further get that sine x plus y is finally equal to sine x cos y plus cos x sine y
 so now we have also derived the expressions for the sign of sum and difference of two angles which is what we did in the previous couple of slides and i have summarized the two equations over here now if we add these two equations this term here and this term here will get cancelled out and will get twice of this term and therefore two times sin x cos y is equal to sin x plus y plus sign x minus y in a similar manner if we compute sine x plus y minus sin x minus y then this terms these two terms will get cancelled out and will get twice of cos x sine y therefore two cos x sine y equals sin x plus y minus sign x minus y will now see how to compute the sign of double of an angle if we simply know the sign and cosine of this angle x is very simple we go back to the expression for sin x plus y and we replace this y with x
 so we get sine two x is sine of x plus x now in this formula we have y equal to

x

so y equal to x here and y equal to x here

so we get sin x cos x plus cos x sin x which is equal to two times sin x cos x earlier we had seen how to find the cosine of the half of some angle

so we saw how to compute the cosine of fifteen by two degrees when we just knew the value of only cosine of fifteen degrees

so similarly we will show you how to compute the sign of the half of some angle x

so we start with this formula cos two x equals to one minus two sine square x which we have already derived on one of the previous slides

so from here by doing manipulation what we get is sin square x is one minus cos two x over two and therefore sin x is equal to plus minus root of one minus cos two x over two

so again here the choice of plus and minus will depend on the value of x

so for example if we were to find out sine of 7.

5 degrees using this formula here we get sine of 7.

5 degrees is equal to now we know that sine of 7.

5 degrees is a positive real number and therefore we only take the positive sign here

so it is equal to square root of 1 minus cos of 15 degrees over 2 and this we can compute since we already know the value of cosine of 15 degrees we already know this equation that cos of x plus y plus cos of x minus y is two cos x cos y now suppose that x is actually equal to the sum of the the half of the sum of two different angles and y is the half of sum of ah half of the difference of these two same angles a and b

so if we substitute x with a plus b by two and y with a minus b by two in this equation what we end up getting is that two cos a plus b by two into cos a minus b by two equals now cos x plus y but x plus y is a plus b over two plus a minus b over two which is a

so we get cos a and similarly x minus y is b

so cos of x minus y is cos of b

so this formula actually gives you a way of converting sum of two cosines into product of two cosines of the sum and differences half of the sum of sum and differences

so for any angle a and b cos a plus cos b which is the sum of the cosine of the angles is two times cosine of a plus d by two times cosine of a minus b by two and similarly if we also remember we had also derived that two sine x sine y equals cos of x minus y minus cos of x plus y and again doing the same substitution here with x equal to a plus b by two and y equal to a minus b by two what we will end up getting here is 2 into sine of a plus b over 2 into sine of a minus b over two equals x minus y is equal to b

so cosine of b minus x plus y is a cosine of a

so this formula again gives you a way of expressing the difference of the cosine of two angles as a product of the sine of the half of the sum and the difference of those two angles similarly we know that sine of x plus y plus sine of x minus y is two sin x cos of y and we do the same substitution here again with x equal to a plus b by two and y equal to a minus b by two

so what we get is two times sine of a plus b over two into cosine of a minus b over two equals x plus y is a

so sine of a plus x minus y is b x minus y is b

so sine of x minus y will be sine of b

so this is another expression for converting these sum of sines to product of sine and cosine similarly we also know that sine of x plus y minus sine of x minus y equals since sine of x plus y is sine x cos y plus cos x sine y

so when we replace this with that expression and also the similar expression for sine of x plus x minus y what we get is

so we can write it here sine x plus y is sine x cos y plus cos x sine y and sine x minus y is sine x cos y minus cos x sin y

so if we subtract sine x minus y from sin x plus y what we get is 2 times cos x sin y

so this expression here and again in this expression we will we can replace x with a plus b by two and y with a minus b by two and what we will end up getting is two times cosine of a plus b by two into sine of a minus b by two equals sine of a because x plus y is a minus sign of b

so this gives you an expression for a way of converting the difference of the sign of two angles to again the product of sine and cosine now let us see how to calculate the cosine and sine of thrice of an angle

so cosine of three x can be written as cosine of two x plus x but we know that cosine of a plus b is cosine of a cos of b minus sine a sine b now using this formula here with a equal to two x

so this a will choose it to be two x and b as equal to x we will get cosine of three x as cos two x times cos x minus sine two x times sine of x which is equal to now we know that cosine of two x is equal to two times cos square x minus one

so we use that result which we had derived earlier times cos x minus and we had also derived that sine of two x is two times sine x cos x

so that is what we are going to use for sin two x here times sine x which is equal to minus two cosine of x

so this is this particular term two times cosine x times sine square x but sine square x is nothing but one minus cos square x

so finally we get two cos cube x minus cos x minus two cos x plus two cos cube x which is equal to

so 2 cos cube x here and 2 here

so that becomes 4 cos cube x minus 3 cos x

so that is cosine of

so this formula helps you to

so if you know the value of cosine of x you can very easily compute the value of cosine of three times x similarly we now derive another expression to compute the sine of three times x sine of three x in terms of sine of x

so again sine of three x we are going to use the formula sine of a plus b equals sine a cos b plus cos a sin b and we are going to use we are going to put a as two x and b as x

so when they get added up we get sine of three x

so therefore we get sine of three x equals sine of a but a is two x

so sine of two x times cos of b but b is x

so cos of x plus cos of two x times sine of x we know that sine of two x is two sine x cos x and cosine of two x is one minus two sin square x

so we are going to use these two expressions here

so this is the expression for sin two x

so we are going to multiply that with cos x plus cos two x is one minus two sin square x therefore two sine x times cos square x but cos square x is one minus sin square x and when then we can open up this expression

so it becomes sin x minus two sin cube x which if you simplify further it will be equal to three sine x minus four sine q of x

so that is your sign of three x

so again as you see we can if you just know the value of sine of x you can find the value of sine of 3x and of course you can do vice versa also because this is actually a if you see this part is cubic in terms of sine x

so if you know the value of sine of three x you can find sign of x by solving

finding the roots of a cubic equation most of this lecture we have been talking and also the previous lecture we have been talking about sine and cosine and giving expressions for sine of x plus y sine of x minus y cos of x plus y cos of x minus y and then sine of two x sine of three x cos of two x cos of three x but then we are going to introduce this other function called tangent of x as you might already be knowing from your trigonometric ratios but we introduce it here more formally now

 so again we consider a unit circle here of unit radius whose centre is at o this point o and this is the the horizontal axis is the x axis the vertical axis is the y axis and suppose we have this point p at on the circle here and whose coordinates are a the x coordinate is a and the y coordinate is b

 so this is a and this is b and therefore the tangent of the angle of rotation because if you see this op radius is initially here it is initially at ob

 so so the ray we initially have a ray o b here and for b to reach this point p we have to rotate this ray around this centre o by an angle of x

 so the amount of rotation is x and therefore this tangent now if you look at this right angle triangle here because this is a perpendicular

 so if you look at the right angle triangle here tangent of this angle is equal to the value of the opposite the length of the opposite side

 so the opposite side to this angle x is this side whose length is equal to b

 so b upon the length of the adjacent side

 so the adjacent side to this angle x is this side o a whose length is a

 so tan x is b upon a and if you try to see further in probably lecture one or yes lecture one we had defined sine of x as the opposite side the length of the opposite side divided by the length of the hypotenuse but since this is a unit circle this hypotenuse is actually a radius of unit length and therefore sine x was simply equal to the y coordinate of this point p and cosine of x was equal to the length of the adjacent side of this right angle triangle which is nothing but the x coordinate of this point p which is a and then we immediately realized that b by a is nothing but sine x divided by cosine of x and therefore we get the relation that for any angle x the tangent of x is equal to sine x upon cosine of x let us try to now plot the graph of tan x on the y axis versus x on the horizontal axis here and for us to do that we will take help of again this unit circle here with center at o

 so lets say that we start with the angle of rotation x to be equal to zero degrees

 so we are somewhere here at zero degrees

 so when x is at 0 degrees the the we are actually somewhere at this point

 so because there is no rotation

 so this op is actually over here and the coordinate of this point is one comma zero and therefore since tan x is equal to the ratio of the y coordinate to the length of the y coordinate of this point to the the value of the x coordinate

 so when x is equal to 0 we are here and then if you see this ratio is simply equal to 0 because at this point the y coordinate is zero

 so therefore at x equal to zero tan of x is zero

 so so this is where we are on the graph at x equal to zero and then as we increase x

 so we what that means is that this ray we try to rotate in the anticlockwise direction

 so that is going to increase the value of x

 so when we increase the value of x the what starts to happen is that the y coordinate which was initially 0 will start becoming will start taking positive values for example at x equal to 45 degree we have a right isosceles triangle here for which both the y coordinate and the x coordinate will be equal right

and therefore when x is equal to 45 degrees tan of x will be equal to one because both the coordinates are one upon root two
 so that point is somewhere here
 so when x equal to pi by four lets say this is one
 so the value of tan x will increase from zero to one like that and then when x goes increases further what happens is that the value of the
 so for example when it when the ray rotates further in the in the anticlockwise direction and suppose the ray is over here now
 so the angle now is more than let us say
 so suppose if this was forty five degrees the angle is now more than forty five degree
 so maybe we are somewhere close to pi by two
 so as you go as the angle moves closer to pi by two what happens is that the x coordinate of the point starts to reduce and it becomes very close to zero
 so essentially the a goes to zero
 so as a goes to zero but the y coordinate is still close to one
 so it will be somewhere approximately one as you go closer to pi by two
 so since this will be finite but this will go to zero the tangent of this angle x as x goes near and nearer to pi by two will start going to infinity because one by zero
 so this is how it will go
 so it will go to infinity and similarly we can draw it ah draw the same thing for the negative side
 so if you if you start from here and if you go in the clockwise direction
 so now we are reduced
 so as we know going in the clockwise direction will give you negative values of this rotation angle
 so this is the
 so for example when we are here this angle is let us say minus forty five
 so that corresponds to x equal to minus pi by four on this graph but when we are on this side the x coordinate of any point let us say in this fourth quadrant the x coordinate is still the x coordinate a will still be positive but the y coordinate is going to become negative and therefore in this quadrant here the value of tangent of x will be negative and if you try it for example if this this angle is 45 degree then what we will see is that tan of minus 45 will be equal to minus 1 because the if you see this point q here the coordinates of this point will be the x coordinate will be 1 by root 2 but the y coordinate will be minus 1 by root 2 because it is on in the fourth quadrant minus 1 by root 2.

 so when you take the ratio of minus 1 by root 2 divided by one by root two you get minus one which is let us say somewhere here this is minus one
 so so somewhere here and then similarly when we go to ah from minus pi by four further clockwise towards minus pi by 2 again what happens is that the x coordinate goes to 0 but since the value of the y coordinate is negative this ratio will go to minus infinity like this and in a similar manner this whole graph can be filled up by continuing to rotate further beyond pi by two this graph can be completed
 so compared to sine x and cos x c sin x and cos x were bounded functions what i mean by bonded is that for any x the value of sin x was always between minus one and plus one and the value of cos x was also between minus one and plus one but that is not
 so with tangent of x the value can really go unbounded
 so essentially for x the angle x when it is actually the equal to odd multiples

of pi by 2 what because if you see this is nothing but tan x is nothing but sine x by cosine of x

 so it is very much clear that tan of x will be unbounded for all those x for which cosine of x goes to zero and we know that cosine of x goes to zero only when x is an odd multiple of pi by two

 so whenever x is an odd multiple of pi by two tan x will be unbounded it will be minus infinity or plus infinity

 so just as we had ah computed sine of minus x and cos of minus x we can also calculate time of minus x but since tan of x is sine x by cos x tan of minus x will be equal to sine of minus x divided by cosine of minus x i mean we can we can just treat this as some other variable y and then tan of y is sine of y divided by cos of y but we know that sine of minus x is since sine is an odd function this is equal to minus sin x and cosine of minus x is because cos is an even function it is equal to cosine of x but then sin x upon cos x is tan of x

 so this is equal to minus tan of x

 so what we see is that tan of x is actually an odd function because tan of minus x is minus of tan x we can find sine of x plus pi again taking this as y tan of y is sine y by cos y

 so it is sine of x plus pi by cos of x plus pi now sine of x plus pi is equal to and you can again use this formula for sine of x plus y

 so it will be sine x into cos y plus cos x into sine y

 so this is sine of x plus pi divided by cos of x plus y is cos x cos y minus sin x sine y now we know that sine of pi is zero therefore this term will go to zero and even here this term will go to zero because sine of pi is

 so what remains is sine x cos pi divided by cos x cos pi which is equal to sine x by cos x which is equal to tan of x

 so what we see is that the tangent function is actually periodic with pi because for any angle x tangent of x plus pi is equal to tangent of x and similarly we can even show that tangent of x minus pi will be equal to tangent of x

 so let us see what tan of tangent of pi by 2 minus x is since we had similar formulas for sine of pi by 2 minus x and cos of pi by 2 minus x now if we treat this as y then tan of y is sin y by cos y

 so therefore this is sine of pi by 2 minus x upon cos of pi by two minus x but from previous slides we had shown that sine sine of pi by two minus x is equal to cos of x and cos of pi by two minus x is equal to sine of x

 so tan of pi by two minus x as is actually equal to the inverse of tan of x which is usually called which is actually a new function that we are defining here its actually called cotangent cotangent

 so its written as co tangent but we write it as cot in short

 so this is cot of x

 so tan of pi by two minus x is just the inverse of tan of x which is also written as cot of x and here in this class in the third lecture today we started with the expressions for cos of x plus y and cos of x minus y and we derived lot of many different different formulas for cos two x cos three x sin two x sin three x and we also discussed the ah tangent ah function we we formalize it as a trigonometric function and we started with some simple expressions for tan of pi by two minus x for example here we show that it is equal to one upon tan x we also drew the graph for tan x and we saw that the tan ah tangent function is periodic with pi in the next class we are going to define the domain and range of this cotangent function and then we are going to come back to the tangent function and just like what we did for sine and cosine we are going to see if we can derive formulas for this sum and for the tan of the sum and difference of angles

so we are going to see if we can write tan of x plus y in terms of tan x and tan y and similarly for tan x minus y and also expressions for tan of 2 x and tan of 3 x in terms of tan of x thank you you