welcome to this second lecture on trigonometric functions in the first lecture we had introduced the background of trigonometric functions all that you had studied in grade ten we introduced two trigonometric functions sine and cosine of x and we had started discussing some of its properties
 so we will continue with that in this lecture
 so we would like to answer the next question which is for what x is cosine of x equal to zero if you recall we had a unit circle whose center was o and let us consider this point p on the unit circle having coordinates a and b
 so the length of this segment line segment o a
 so this point is a here is a and this is of length b and we know that cos of this angle of rotation the cosine of that angle of rotation is a
 so what we are if we are trying to find out angle x as that cos of x is equal to zero since cos of x is equal to the x coordinate of that point what we are essentially looking for is those angles of rotation for which the x coordinate of the final point after the rotation is equal to zero
 so on this circle there are two points for which the x coordinate is equal to zero
 so one is this point here
 so this is the x axis and this is the y axis
 so at this point the x coordinate is zero and then the other point is this point which is zero minus one
 so these are the two points where the x coordinate is zero now this point here corresponds to an angle of rotation
 so we start if we start with this ray here then we reach this ray here reaches here if we rotate it by a quarter of a revolution which is 90 degrees or pi by 2 radians
 so therefore one solution is that x is equal to pi by two radians and the other solution is when you reach this point
 so this point corresponds to three quarters of a revolution and three quarters of a revolution is 3 pi by 2 radians
 so that is the other solution and as we have seen both sine and cosine of x repeat their values after every integer multiple of two pi
 so cos of x is the same as cos of cos of x is the same as cos of x plus k times two pi
 so therefore the solution to this equation cos x equal to zero is when x is equal to n plus half times pi where n is integer let us try to find out the sine and cosine of some angles that we come across often let us focus on this right angle triangle here abc where this angle is 90 degrees and this angle is theta of course this third angle then is pi by two minus theta
 so what we will see here is that cos of theta is equal to the length of the segment a b by ac and sine of pi by 2 minus theta is equal to now we are trying to look at the other angle which is this angle pi by 2 minus theta now the sign from the definition of sine of an angle the sine of this angle will be equal to the opposite of this angle
 so the opposite of this angle is this side a b divided by the hypotenuse which is ac
 so what we see here is that these two ratios are the same and therefore cos of theta is equal to sine of pi by two minus theta
 so therefore if you know the sine of any angle you can also know the if you know the sign of each and every angle you can know these cosine of each and every angle
 so essentially essentially they are one and the same and this is the relation between them well let us try to find the cos and sign for some commonly some angles which we come across commonly

so let us think of this isosceles right triangle triangle abc where this is 90 degrees and this is an isosceles right angle triangle therefore a b is equal to b c is equal to one unit and because it is isosceles that is this side and this side are of equal length this angle and this angle will be also equal and therefore both of them will be 45 degrees each which is both of them will be pi by 4 radians and by the pythagoras theorem the length of this hypotenuse will be square root of a b square plus b c square which is equal to square root of two units and therefore cos of this angle pi by four will be equal to adjacent divided by hypotenuse which is one divided by square root of two and in a similar manner sine of pi by 4 will be equal to opposite divided by hypotenuse which will also be the same

so when the angle is equal to pi by 4 or 45 degrees both cosine and sine of that angle are one and the same and they are equal to one upon square root of two let us take another little example where we would like to find out the sine and cosine of pi by 6 radians which is 30 degrees

so we have a triangle right angle triangle here abc where this angle is pi by 6 radians or 30 degrees and we would like to find the sine and cosine let us extend this line cb the straight line cb like this and let us make another angle here which is equal to minus pi by six

so we construct this another ray here such that this angle is the magnitude of this angle is also pi by six ok now this ray and this straight line are going to intersect at this point let us call it d and now we focus on this triangle a c d but before that what we see is that if we if we just ah look at these two triangles a b c is one of the triangles and the other triangle is a d b

so this triangle and we realize that both these triangles are congruent because they have a common side a b and this angle is 90 and because this line c d is a straight line this angle is also 90 and then of course by construction this and this angle this angle and this angle are also equal and therefore triangle abc and triangle a b d

so triangle a b c and triangle a b d are congruent or exactly the same and therefore the length of the sides are also equal

so suppose if this ac is equal to one unit then a d is also one unit because these two triangles are congruent further let us focus on the bigger triangle a d c

so i am now i am talking of this triangle a d c this particular triangle we see that by the congruency of these two triangles this angle and this angle have to be equal

so if if the measure of this angle is theta then this angle is also theta and this total angle here is pi by three or sixty degrees

so what we see now is that if you look at this triangle a d c its an isosceles triangle to start with because this angle and this angle are equal and therefore since the sum of all the internal angles of a triangle is 180 degree this 60 degrees which is pi by 3 plus theta plus theta

so pi by 3 plus theta plus theta has to be pi radians and which actually implies that theta is equal to pi by three radians

so this theta is also pi by three this angle is also pi by three and this of course is pi by three

so this triangle adc is an equilateral triangle it is an equilateral triangle because all the three angles are equal to pi by 3 radians or 60 degrees and therefore the length of this line segment c d will also be equal to the the other length of the other two sides which is one unit

so this c d is also of length one unit further

so what we have now is that the length of c d is one unit further because these two triangles a b c and a b d are congruent the length of these two sides b c

and b d

so this length and this length have to be equal

so if the whole length c d is one unit then it turns out that this length has to be half a unit this has to be half a unit and therefore now we can say that the cosine of this angle pi by six sorry this sign of this angle

so sine of this angle pi by six is equal to

so sine of pi by 6 will be equal to

so sine of this angle

so let us focus on this triangle a b c whose hypotenuse is of length one unit and c b is equal to half unit and therefore sine of pi by six will be opposite by hypotenuse which will be half divided by one which is equal to half

so through this simple construction we showed that sine of pi by six is equal to half and similarly because in the previous class we had shown that sine square x plus cos square x is one

so using that relation what you can show is that cos of pi by six will be equal to root three by two another question that would come to mind is is there any relation between sine of x and sine of minus x and similarly between cos of x n cos of minus x

so we have again drawn a unit circle here with center at o this is the x axis this is this is the x axis this is the y axis and we have a point p here whose x and y coordinates are e and b respectively and this angle of rotation is x

so therefore if i drop a perpendicular from this point p onto the x axis at this point a then this length o a will be equal to a

so this o a will be equal to a and this this length here will be equal to b now let us rotate since we are interested in this angle minus x then we need to to get minus x we need to rotate this particular radius in the clockwise direction by the same amount of rotation that we did for this angle x

so when you rotate it by the same amount this angle is minus x and when we rotate starting from here when we rotate in the clockwise direction by the same amount of rotation as we did when going from here to here let us we say that we reach a point q to find sign of minus x find sign of minus x sin of minus x will therefore be equal to suppose that the coordinate of this point q is c and d then sine of minus x

so sine of x is equal to b that is something we already know sine of minus x will be equal to

so this is opposite

so that is the y coordinate of ah this point q which is d divided by the length of the hypotenuse which is equal to one

so sine of minus s is equal to d

so essentially we need to see if there is any relation between this d and b now let us see this ah triangle

so this point is a if we see these two triangles here

so one triangle is o a b

so that is this triangle here and the other triangle is o a q then what we see is that sorry

so between these two triangles they are congruent because

so triangle o a p is try is congruent to triangle o a q and that the reason being that of course this side o is common to both of the both of them this side op of this triangle oap is equal to the length oq of the triangle oaq because both of them are radius of this unit circle

so we have two sides which are equal and then this angle x here which is the angle aop for this triangle is equal to this angle because both of them are the same in magnitude

so therefore these two triangles are congruent now this when we had actually

drawn this this point a we had we had dropped a perpendicular from this point p to the x axis

 so this was 90 degrees now because these two triangles are congruent this angle should also be equal to this angle which will also be 90 degrees and therefore what we see is that this p a q will actually be a straight line because this angle is 90 and this is 90 therefore the total angle here this total angle is 180 degrees and therefore this p a q is a straight line which is bisecting which is basically intersecting with the x axis at 90 degrees and therefore it is obvious that the x coordinate of this point q will also be equal to a

 so c is equal to a therefore that shows that because that is because of the fact that this whole line is a straight line and it is intersecting with the x axis at 90 degrees

 so essentially this line segment here is parallel to this coordinate axis the y coordinate axis because these two lines are parallel the coordinate c here of this particular point will have to be equal to will be equal to this a here right

 so therefore c is equal to a but what about d now because these two triangles here are congruent the length of this side of the first triangle will be equal to this will be equal to the length of this side of triangle oaq therefore this length the magnitude of this length will also be equal to b but since this is in the fourth quadrant this is on the this is below the ah x axis

 so therefore d will be equal to minus b from where we conclude that now from here and here we therefore conclude that sine of minus x will be equal to minus b which is equal to minus of sine x which follows from the fact that b is equal to sine x to start with and therefore what we see is that sine of minus x is equal to minus of sine x and this is a very fundamental relation now these type of functions where f of minus x is equal to minus of f x have a special name and they are called odd functions they are called odd functions if we use the same figure here then what we can see also is that cos of x if you look at this triangle o a p then cos of x is equal to this length a divided by one which is a and cos of minus x is what for minus x we look at this triangle o a q and cos of minus x will then be equal to the same a divided by hypotenuse which is of length one therefore this is also equal to a and therefore cos of x and cos of minus x are always equal and such functions where f of x equal to f of when f of x if there is a function f such that f of x is equal to f of minus x for all x

 so that is not just for one value of x but for all values of x

 so here also this if it has to be for it to be called as an odd function the function must satisfy this relation not just for one value of x but for all values of x in its domain

 so we see that cos x is equal to cos of minus x for all values of x belonging to real number the set of real numbers which is the domain of the cos function

 so such functions are said to be even functions they are said to be even functions

 so next we try to delve a little deeper or dig a little deeper into the range of values of sin x and cos x as we move as we increase x from zero to two pi

 so when we are between when this angle x the rotation angle x is between zero and

 so 0 is when you we are here and as you as we move this point on the circle in the anticlockwise direction till we reach this point all we are all we are always in the first quadrant

 so when x is between 0 and pi by 2 radians then the point p is in the first quadrant and since sin x is equal to b the y coordinate of the point and cos x is equal to the x coordinate of the point now in the first quadrant the x coordinate as you can see here is between zero and one and therefore cos x will

be between it will lie in the interval zero to one

so cos x will be greater than zero and less than equal to one whereas sine x which is the ah y coordinate of this point will also lie in the interval zero to one

so i put a curly bracket here because i have defined the first quadrant to be x less than pi by two

so sin x will has to be less than one because ah sine x is equal to one only when x equal to pi by two and therefore it will never attain this value one and therefore there is a round bracket here and similarly we can fill up the other entries of this table for example if as we move along the circle if we move further from this point in the anticlockwise direction then we are in the second quadrant

so that is when the rotation angle is between pi by two

so pi by two is this much all the way till pi

so pi is half a rotation

so in the second quadrant sine x is basically if you see sin x is the y coordinate right

so sine x will again lie between it is on the positive side of the ah its on the upper side of this horizontal x axis

so the y coordinate of any point in the second quadrant will always be between zero and one

so this will also lie between but in this case it will lie between zero and one but for the cos x for the cosine in the second quadrant what happens is that point is on the other side of the of this y axis

so what happens is that the x coordinate becomes negative and since cos sine of an angle is equal to the x coordinate of the corresponding point on the circle because the value of cos x in the second quadrant will go from

so when we are here cosine of this pi by two is actually zero and when we reach at this point this is equal to this much this coordinate of this point is minus one comma zero

so the x coordinate is minus one

so cosine of one hundred and eighty degrees is equal to minus one

so in the second quadrant cosine of x will lie between minus one and zero and in a similar manner ah the other entries could be filled up

so basically we have to keep moving in the anticlockwise direction starting from here to when we move further then we are in the third quadrant till this point and then from when we move further from this point back to where we started from then we are in the fourth quadrant now let us try to plot the graph of the sine function

so on the x axis we have the angle of rotation x on the y axis we have the value of sine of the angle of rotation x

so let us say this is one and this is minus one and let us say that now i have drawn a little circle here of ah unit radius with center at this point o and let us say that we start from this point here start from this point here and try to move in the anticlockwise direction now at this point we know that first of all from the previous slides we know that sine of x of any point will be equal to the y coordinate of this point now at this point when x at this point there is no angle of rotation there is no rotation therefore the angle of rotation is zero

so on the x axis we are here x equal to zero and since we are at this point here on the circle the y coordinate is zero and therefore sine of x equal to zero will be zero

so we draw this point as we as we move further in the anticlockwise direction let us say we reach half way between this position and this position which is

here
 so we are somewhere there
 so this has to be half of 90 degrees which is pi by four or forty five degrees
 so when we reach here the sign of this angle will be equal to the y coordinate of ah this point which will be equal to one by root two which is approximately zero point seven zero seven
 so we see here this is pi by 4 and the value of sin pi by 4 is 0.
707 and
 so since this is 1 and
 so half of that will be
 so this this will be 0.
5
 so let us say something approximately this much sorry sorry this will be 2 by 3
 so that is 0.
66
 so this will be something like
 so this length will be something like from here to here because one is shown to be three little squares here
 so sin pi by four will be approximately one ah zero point seven
 so therefore when you go from zero to pi by four when you plot this graph of sine x it looks something like this and then when we further go in the anticlockwise direction by another forty five degrees which we reach this point whose coordinate is whose y coordinate is equal to one and this angle of rotation now is pi by 2 therefore sine pi by 2 is 1 and therefore we reach this point
 so now we connect the graph like this then further anti clockwise direction starting from this point and then going in this direction we are in the second quadrant but now the value of the y coordinate on any point here in the second quadrant has to be less than one because we are coming down here
 so sine x will again start decreasing from one and till we reach this point here
 so for this point here the total rotation angle is a straight line which is 180 degrees or pi radians and the coordinate of this point here is minus one zero
 so the y coordinate of this point is zero and therefore sine of one eighty degrees is zero and hence in the in the second quadrant if we try to plot this graph it will look something like that
 so so this point here corresponds to this point here on the graph sine of pi is equal to zero and then we can further move ahead and we just have to for any point here for example here let us say this point here we just need to look at the total angle of rotation starting from here and then corresponding to that angle of rotation we get this point and we then just need to look at the y coordinate of this point and that y coordinate has to be plotted on the y axis here that is how we complete this graph
 so if you try to do it further at three pi by two which is at this point sine of three pi by two will be equal to minus one
 so you should be somewhere here
 so if you try to connect it you might get a graph something like this and then
 so so
 so going from pi to 3 pi by 2 is when you are in the third quadrant and then further if you go then you are in the fourth quadrant and your curve will look something like that
 so this is how you plot ah sine of x cosine of x can be plotted in a similar manner its just that instead of ah looking at the y coordinate of this points you have to plot the value of x coordinate of each of these points here on the y

axis
 so that is how you get the graph for cosine of x we are not going to us to ourselves suppose that you have two angles x and y and you know the values of sine x sine y cos x cos y
 so can you find the value of this angle x minus y can you find cosine of x minus y and then maybe cosine of x plus y or cosine of x plus two y sine of x plus two y or sine of twice x
 so this is what we are going to address next we are going to derive formulas for expressing cosine of difference and sum of angles in terms of cos x sine x cos y sin y and suppose that
 so o is the center of this unit circle and consider this point q here
 so let me use a blue pen also
 so let this angle of rotation be x and then we have another point p and let us say that the angle of rotation for this point p is y
 so the coordinates from the definition of sine and cosine of x and y the coordinate of this point q will be the x coordinate will be cos of x and the y coordinate will be sine of x for this point p the x coordinate will be cos of y and the y coordinate will be sine of y and then of course this angle here will be equal to x minus y x minus y and now we also draw another point are such that the angle of rotation of for this to get from here to r is equal to
 so this angle in red is also equal to x minus y
 so that is also x minus y
 so now we have and let us say that this point here is equal to a
 so this point a has coordinates one comma zero the coordinates of this point r will be because the angle of rotation for this ah point r is x minus y in red
 so the coordinates are going to be cos of x minus y is the x coordinate the y coordinate is sine of x minus y let us now focus on two triangles
 so we are also going to
 so we will first look at the triangle o p q
 so let me join p and q here with this green dotted line
 so one of the triangles is triangle o p q the other triangle to be considered is o a r
 so triangle o a r
 so for that we need to join together a n r now if you look at these two triangles then what we see is that in triangle opq this side oq of this triangle is equal in length to the side o r of triangle o a r because both oq and o r are the radius of this circle of this unit circle further side op of this triangle opq is also of unit length because that is another radius
 so this o p of this triangle o p q this all this o p is also of unit length that is also equal to o a because of this triangle o a r
 so if you see this triangle o this is point a a and then r
 so this o a is also radius
 so therefore o p is of triangle o p q is equal to the side o a of triangle o a r and further angle p o q of this triangle o p q is equal to angle a o r of the triangle o a r o a r because both of these angles are equal to x minus y and therefore these two triangles are congruent they are congruent now using this fact that they are congruent since they are congruent the length of all their sides corresponding sides should be equal and therefore the length of this side q p which is shown by the green dotted line of this triangle o p q must be equal to the length of the side a r of the triangle o a r this is because these two triangles are congruent now we will try to use that this fact now further now this this line ah this length q p is nothing but the distance between the points q and p where the point q has coordinates cos x sin x and the point q has coordinates cos y and sin y i mean writing q p equal to a r is same as writing q

p square is equal to

so if the two lengths are equal their squared lengths are also equal now q p square will be simply equal to cos x minus cos y whole square

so so this is q p square is equal to cos x minus cos phi whole square plus sine x minus sin y whole square right

so that is q p square and that has to be equal to a r square what is a r square now we know the coordinates of the point a and point r the coordinates of point a is one zero the coordinates of point r is cos x minus y and sin x minus y therefore the squared equilibrium length of this line segment a r will be equal to cos x minus y minus one whole square plus sine x minus y minus zero the whole square which will be sine square of x minus y

so these two are equal

so let us try to further simplify them in the next slide the first expression cos x minus cos y whole square plus sine x minus sin y whole square equals

so the first square equals cos square x plus cos square y minus two cos x cos y and then plus the second square equals sine square x plus sine square y minus 2 two sine x sine y but then we know that for any angle x sine square x plus cos square x is equal to one therefore these two get added up and become one plus these two also get added up and become one minus two cos x cos y minus two sine x sin y and this was equal to the the

so that was the simplification of the first expression this one and that is equal to this particular term ah this particular expression which is the second expression

so let us expand that also

so we said that this has to be equal to cos x minus y minus 1 whole square plus sine square of x minus y which is equal to cos square x minus y plus one minus two cos x minus y plus sine square of x minus y which is equal to now this cos square x minus y and sin square x minus y will add up to one

so this will simplify to one plus one minus two cos x minus y now since since this and these are equal

so when you equate when we equate them what we end up getting is that cos of x minus y is equal to cos x cos y plus sine x sine y and this is a very fundamental result which we will be using in our ah other lectures later on ok

so we just now we saw that given any two angles x and y cos of x minus y equals cos x cos y plus sin x sin y how about cos of x plus y we can use this formula for cos x minus y to also derive an expression for cos x plus y as follows we can write this as cos of x minus of minus y and then use this formula

so this will become using this formula this will become cos x into cos of minus y plus sine x into sine of minus y which is equal to cos x now we had shown that cos is an even function therefore cos of minus y is equal to cos y

so we have cos y here but sine of y is an r function and therefore sine of minus y is minus sine y and therefore we get a minus sign here and it becomes minus sin x sine y

so with this we finish the second lecture where we started with more relations between sine and cosine we showed that the sine function is an odd function the cosine function is an even function we also showed how to plot the graphs for sine and cosine and finally we also derived an expression here for the different the cosine of the difference and the sum of two angles in the next class we will discuss the sign of how to derive ah basically starting from these equations itself we will derive the sign of difference and sum of two angles the sine and cosine of twice and thrice of angles and some other relations you