welcome to the first lecture on trigonometric functions we will start with discussing a little bit of background which you would have already studied in your 10th standard it is basically a greek word which is composed of trigono and metron

so trigonomeans triangle metronome means measure

so essentially the study of ah measuring the sides of triangles and also finding the relation between the angles and sides of triangles for example let us take ah this right angle triangle here abc

so this is 90 degrees let this angle be theta you would have studied in your trend standard that where you would have defined your trigonometric ratios for example cos theta sin theta and tangent of theta

so for this angle theta we say this side a b as the adjacent side and this side ac which is opposite to the right angle in the right angle triangle is called the hypotenuse and this side for this angle theta this side a b is called the adjacent side and the side opposite to this angle theta will obviously be called the opposite side and if you recall cosine of theta which you write as cos theta

so cos is actually ah sort of short form for cosine

so it is actually written as cosine and cos is the short form for it

so cos theta you have already read this that it is actually equal to the length of the adjacent side which is length of the segment a b upon the length of the hypotenuse which is length of ac sine theta again s i n is a short form for sine s i n e sin theta is equal to the length of the opposite side which in this case is the line segment b c divided by the length of the hypotenuse which is ac and then finally tan theta which is the tangent of this angle theta is equal to the length of the opposite side which is b c upon the length of the adjacent side

so this is already what ah has been studied by you just recap these trigonometric ratios have been used by you to solve various problems

so one of them being problems related to heightened distances for example this particular problem here ah where you have a tall building here whose height is unknown to you let say that the height is h meters and when you stand up here at this point a and look at the top of the building the elevation angle that you measure with respect to the ground a b is 30 degrees when you move towards the building to another point c by 10 meters

so this distance ac is 10 meters and again look up to the top of the building the elevation obviously increases to lets say 60 degrees and then you are of course asked to find the height or estimate the height of the building based on this measurement of these elevation angles

so the way you do it is

so let h denote ah the height here let the distance b c b equal to let us say s meters and then you use the tangent formula

so if you use the tangent formula for this for this angle which is 30 degrees what you get is tan of 30 degree is equal to h upon s plus 10 because h is for this 30 degree angle h is the opposite side and s plus 10 here is the adjacent and of course you know that tan thirty is equal to one upon root three and then you look at the other triangle c d b this triangle and you again write the tangent of the angle the 60 degree elevation angle and using the same formula you get it to be equal to s divided by s in this case tan of sixty is equal to root three

so now you got two equations and two unknowns you should be able to find both h and s there is a little ah question here

so in the previous page when we were discussing this hypotenuse opposite and adjacent side the definition of which side is opposite is very clear because it is the side which is to the opposite ah of this angle theta

so for example if i were to consider lets say the same right angle triangle

again but instead of considering this angle if theta were to be if theta were to be the other angle angle a c b if this were to be theta then the definition well the definition of hypotenuse would still remain the same because hypotenuse is the side which is opposite to the right angle
 so this will still remain to be the hypotenuse but the adjacent and opposite sides will now change now this side b c will be
 so this side b c will be the adjacent side and this side a b will be the opposite side because now this a b is the side which is actually opposite to this angle now that being the reason
 so if you would have seen ah angles or something which you will encounter everywhere
 so a natural question which comes to mind is how to measure these angles
 so one common measure that we know is degrees but before that let us define what our angle is now consider this ray o a
 so this is a ray and let us say that we rotate this ray about this point o
 so this will keep this point o as fixed and then we will sort of
 so we will keep this point of fixed and then we will move a like this
 so this fixed point will be called the vertex and suppose we move it anticlockwise and lets say that this is
 so this tip here moves from this lets say this point a to this new point b
 so this length and this length is the same and we just rotate it like this ok now this side o a is usually called the initial side of the angle
 so when you rotate it
 so angle this this angle is nothing but the measure of how much rotation you do it is a measure of how much rotation is performed when you go from this position to this position
 so this side o is called the initial side of the angle this side ob is called the terminal side when the rotation is anti-clockwise as in this example here the the angle is said to be positive and if the rotation
 so so this is let us say for example i let me draw another one here now instead of rotating this o a anti clockwise if we rotate it clockwise
 so let us say we go from here to here
 so this is the initial side this is the terminal side
 so we are rotating it now clockwise
 so in that case the convention usually is that this angle will be negative now to measure angles there are two popular measures of angles one is called one is usually one way to measure it is in in terms of degrees the other way to measure it is in terms of radiance
 so we will discuss degrees first because that is something which is which you are you would have already studied
 so if you we start with one complete revolution
 so again we consider this ray o a and suppose we rotate it like this
 so we take it all the way and then we bring it back
 so one complete revolution
 so one complete revolution is said to be 360 degrees now that there is no math behind you know why this is called 360 could have been called 450 or 800 or 720 reasons as we know now are primarily historic
 so we will stick to ah three sixty and then of course
 so one complete revolution is 360 degrees and of course 1 degree will be equal to 1 by 360th part of a complete revolution ok
 so that is how one degree is defined
 so one degree is essentially the one by 360th part of a complete revolution now let us again look at that ray o a now if you if you see if if one complete revolution is going to be called 360 degrees then if we only do one fourth of a

revolution

so for example if we go from this o a which is actually lying down horizontally to lets say o b which is standing upright which is standing upright and if you look at this angle here then if you if you see if you rotate this ob again by another

so let us say that this angle is equal to theta

so we rotate this starting from ob if we again rotate this ray in the anticlockwise direction by another theta then we should be exactly lying down again right but in the opposite direction compared to o a

so it will look something like this let us say c

so this also has to be theta and then another claw anticlockwise turn starting from oc and again by theta should take us here

so this is another theta and then again another anticlockwise term by the same angle theta will take us back to where we started from originally which is o a but then what we see is that fourth time because if you add all these angles then four times theta is exactly equal to one complete revolution which we have seen the last page it should be equal to 360 degrees and therefore this angle theta is one-fourth of 360 which is 90 degrees and that is why if you if you change from a lying down position to straight up right the amount of rotation that this ray would have suffered is 90 degrees in a similar manner lot of other ah you can define all other angles like 180 degree

so 180 degrees will be if you go back to the ah we go back to the last slide if you start from o a and if you take 2 90 degree rotations

so for example if you go from o a to o b

so that is taking a 90 degree rotation once and then another 90 degree rotation from ob to soc and what you see is that this oc is exactly is exactly like is a lying exactly down and is exactly opposite to this o a

so essentially this c a is a straight line

so c o a is a straight line and the angle of rotation is this 90 plus this 90 and that is 180 degrees

so that is why we we usually say that a straight line is 180 degrees another measure of angles is called the radian and that might be new to some of you

so the way it is defined is let us look at the the circle here or whose centre is at this point o and whose radius is one one unit

so this is a unit circle ok and then consider this ray o a

so this is a radius o a length is one unit now when you suppose we start rotating it in the anticlockwise direction and just focus on the amount of distance moved by this tip here

so for example if you move this o a slightly then this this ray would look something like this and the tip comes here

so this is the distance traveled right now if you keep on increasing the rotation angle the length of this arc will increase right

so if we keep on increasing up till the point let us say

so we started with the tip was at point a we move to a point b such that the length of this arc a b is also one unit which is equal to the radius of this unit circle right

so when that happens the angle of rotation then is said to be one radian

so a natural question that arises in the mind of a student is what if suppose i have a radius o c here and now i suppose rotate it from oc to od by an angle of two radians and how much is the length of this arc c d of course it it it appears that of definitely it won't be one unit probably more than one unit but how much exactly that is not very difficult to see because this angle two radians is nothing but it consists of two successive repetitions of rotations rotating the ray by one radian two times

so suppose we initially start with the tip of the ray at this point a and let us say that we first rotated by one radian

so you reach this point let us say b and as per the definition what we have seen is that because this angle of rotation is one radian and obviously the length of this arc over here this arc a b this length starting from here till this point b is also one unit but we are looking to find the length of the arc when the angle subtended at the center by that arc is two radians

so what we do is we then move ahead from b and we rotate further by another by another one radian

so we start like this and then we again rotate by one more radian

so we finally reach let us say some point c over here

so this might not be looking exactly as one radian but let us assume

so so when you reach this point c over here start

so starting from this point b to point c if you look at this particular sector o b c and if you look at the sector o a b

so which is this sector exactly they are exactly identical and therefore this length b c should also be equal to one unit because again the angle of rotation starting from ob to oc going from ob to oc is one radian therefore this length of this arc bc should also be one radian and then we of course get the answer because if you now find out the answer to this question where we were asking you to find out the length of this arc cd which subtends an angle of two radians at the center of the circle here what we have is an arc ac

so i am talking of this arc ac where the angle subtended at the center is 1 radian plus 1 radian that is 2 radians and you see that the length of the arc is this one radian plus this sorry this one unit plus this one unit

so its a total of one unit plus one unit is two units and therefore if the angle subtended by any arc at the center is two radians then the length of that arc will be two units right

so it appears that if you increase if you double the the angle of rotation then the length of the corresponding arc will also double

so therefore it has a little table if the arc length if the ah let us say the arc length is one unit then the angle subtended at the center is one radian or vice versa if you increase the angle subtended at the center to two radians and as we saw in the on the previous slide the length of the arc will double from one unit to two units if the angle subtended at the center is for example any fraction or decimal number any real number like three point one seven radians then the arc length will be three point one seven units now we know that for a circle here of let us say radius one unit if i start from this point a

so look at this o a and i make a complete revolution which is that i i go like this and then come back to a if you make a complete revolution then the arc length will be equal to two times pi units and therefore going by this fact that the arc length and the angle subtended are equal what i mean is that if the arc length is one unit then the angle subtended at the center is one radian and if you double it then the angle subtended at the center also doubles then by going by that if the arc length increases from one unit to two pi units then it should be clear that the angle subtended at the center by this complete revolution should be equal to two pi radians and therefore this shows that one complete revolution is equal to two pi radians

so this is something to be remembered that one complete revolution is equal to pi radians here of course pi as all of you know is a constant it is a universal constant and it is equal to the ratio of the circumference of a circle divided by the diameter of the circle

so you take any circle however small or however big in this universe if for that same circle if you calculate the circumference you find out the diameter of

the same circle and if you divide the circumference by a diameter no matter how big or small or whichever circle you draw the ratio is always a constant and that constant is called pi another related question that might come to mind is
 so this let us say that this inner circle here the
 so i have what as you see in this figure there are two concentric circles
 so one is this ah circle with the smaller radius and the other one the outer circle has a bigger radius and both of them have the same center at this point o and let us say that we have a ray here this particular ray and we rotate it in the anti clockwise direction by one radians
 so that the ray now comes here then for the inner circle whose radius is one unit we know that the length of this arc will also be equal to the radius which is one unit right but let us say that this outer circle has a radius of r units
 so this is what i am talking about and it also rotates we rotate this particular ray by one radian and we want to see the length of the arc on the outer circle which is this length x of course that will not be one unit because one unit was the length of the arc in the inner circle and as you can see it appears that this is definitely more than one unit but how much is it
 so if you see in this table for the outer circle let us say that we do not know this arc length
 so for the outer circle if this arc length is equal to x units then as we have drawn it here the angle subtended at the center is is one radian okay and from the previous from the previous slide we know that one complete revolution is equal to how many radians one complete revolution is equal to two pi radians right
 so this is one complete revolution
 so if you if the angle subtended by an arc at the center is two pi radian that corresponds to one revolution
 so if you increase the angle subtended by the arc from one radian to two pi radians which is one complete revolution then the arc length should also increase in the same ratio which is that if since here you are increasing the angle by two pi times the arc length should increase from x to two pi x it should increase from x to two pi x but then we know that the arc length for a full complete one complete revolution is nothing but the circumference of the outer circle which is actually two times pi times r and that should be equal to two times pi times x because that is what we got from this table and therefore this x should be nothing but equal to r units
 so therefore what we see is that we could also define one radian very generally as instead of just looking at a circle of unit radius in general if we have a circle of radius r and if we rotate let us say this look as let us look at this radius here on the outer circle radius is r let us look at this radius this ray now if we rotate it by one radian we reach here
 so in that case because we rotate it by one radian this particular arc length will be equal to the radius itself which is r
 so essentially one radian could be defined as the rotation angle such that the length of the arc corresponding to that rotation angle is equal to the radius of the circle the next step is
 so this is what we were discussing in the previous slide that if you have a circle of radius r and you look at this ray o a if you rotate it by one radians and the length of this arc here will be equal to the radius which is r units
next question is suppose we have a ray o c and i rotate it by theta radians and what is the general formula for the length of this arc c d
 so again we have a table here we know that from the previous slide that if for in this circle of radius r if the angle of rotation is one radian then the arc length will be r units equal to the radius if it is two radians then of course

the arc length will also double it will become two r units in general if it is any real number times one radian for example three point nine at radians then the arc length will also increase proportionately to 3.

98 r and therefore if the angle is in general some theta radiance for example like this here then the arc length should also increase because if

 so what is happening is that compared to one radian we are increasing it or decreasing it to theta radians

 so there is a multiplying factor of theta here going from one to theta and therefore the arc length should also increase proportionately from r to theta r and therefore that is how we get our answer that the length of this arc corresponding to this angle of rotation theta will be equal to theta times r units as many of you would have guessed by now that there is a relation between these measures which is what we will discuss now

 so we earlier said that one complete revolution is 360 degrees that is when we were defining what a degree is and later on we also said that one complete revolution is equal to two pi radians and therefore since they have to be the same two pi radians should be equal to three hundred and sixty degrees and therefore one radian should be equal to three sixty divided by two pi degrees

 so if you use the approximation that pi is equal to twenty two by seven which is an approximation then you just get 360 divided by 44 by 7 which is approximately 57.

27 degrees

 so this formula here 1 radian equal to 360 by 2 pi degrees will give you a conversion from radians to degrees

 so for example if i ask you how much is pi by 4 radians is equal to how many degrees

 so its very simple since 1 radian is 360 by 2 pi pi by 4 radians will be pi by 4 times 360 upon 2 pi degrees which is going to be equal to 45 degrees

 so that is how you convert from radians to degrees and then of course somebody can ask you ok if i give you an angle in terms of degrees how would you convert it into radians

 so for

 so that is also very simple again

 so we just inverse invert invert the whole argument and say that now three sixty degrees is equal to pi radians and therefore one degree has to be equal to two pi upon three sixty radians and suppose if somebody ask you to lets say find out how many radians is three hundred and one hundred and thirty five degrees its very simple because if since one degree is two pi by three sixty radians one thirty five degrees will be equal to one thirty five multiplied by two pi upon upon three sixty radians which will be equal to three pi upon

 so three pi divided by four radians

 so the conversion is very simple let us take a little example here

 so what i have drawn here is a clock

 so as you can see 12 o'clock 3 o'clock 6 o'clock 9 o clock and it is said that the minute hand of the clock is of length equal to five centimeter has a length equal to five centimeter now how much does the tip move how much does the tip of the minute hand move in forty two minutes

 so let us assume that the minute hand was at this position to start with and then it has to rotate by some angle and then finally after 42 minutes it would reach here

 so let us first try to find out this angle of rotation now we know that one complete revolution is equal to two pi radians but then this is not one complete revolution right this is only

 so since in this case one complete revolution of the minute hand of the watch

is going to be one r which is actually equal to 60 minutes whereas we know that in this problem we are asked to find out how much the tip has moved in 42 minutes and because 42 is less than 60 it is not a complete revolution in fact it is equal to forty two upon sixty of one revolution now since one complete revolution corresponds to a rotation of two pi radians forty two by sixty of one revolution will correspond to forty two upon sixty times two pi radians which is equal to one point four pi radians and then how do we find the length of this arc here because the question is asking you how far does the tip of the minute hand move in 42 minutes

 so to find the length of this major arc corresponding to this rotation by angle theta as we saw on the previous slide the length of this arc will be l equal to the angle of rotation theta times the radius of this circle in this case the radius of the circle is nothing but equal to the length of this of this minute hand which is five centimeter

 so the answer is equal to one point four pi times five centimeter and if you use pi equal to twenty two by seven as an approximation then you will get it to one point four times twenty two by seven times five centimeter which is going to be equal to 22 centimeter

 so this was a ah quite a bit of background which many of you would be knowing now the purpose of this session and the other sessions to come would be to generalize these trigonometric ratios that you would have already learnt in your previous classes two trigonometric functions

 so we go back to sine and cosine again and try to generalize them to sine and cosine functions

 so in this slide what we have here is a unit circle it has radius one unit whose center is at this point o this i call this horizontal axis as the x axis and the vertical as the y axis now consider this point p on the unit circle here whose x and y coordinates are a and b respectively

 so what this means is that if you project this point on to the x axis then this length is equal to a units this is a units and this length is the projection of this point on to the y axis and of course that will be equal to b units and let us connect this point o to this point p

 so if we see this

 so what we have here is a right angle triangle and let us call this angle as x and now we are ready to formally define these functions sine of x and cos of x

 so sine of x will be equal to as you have already studied earlier if you look at this triangle o p and let us call this point as b sine of x is equal to b divided by the length of the hypotenuse but since it is a unit circle this hypotenuse is of unit length

 so sine of x is simply equal to this the y coordinate of this point p and similarly cosine of x will be equal to a by the hypotenuse which is again of unit length

 so it is simply a

 so cosine of x is simply equal to the x coordinate of this point which is a of course now that we are defining these two functions sine and cosine we need to define the range and domain of this function if you see this x this x is real valued it can take any real value and therefore the domain of this function both these functions cos and sign is the set of real numbers r

 so the domain is equal to the set of real numbers r and let us talk about the range now if you see for example for the sine function sine of x for this x for any x is equal to the y coordinate of this point p now as this point p moves

 so if you start with let us say suppose if we initially say that p was here then when p is here x is equal to zero and then as you move let us say in the anticlockwise direction on this circle x starts to increase and you can keep on

going like this
  so as you go you will get different values of x and for each value of x you can actually measure the x and y coordinate because for each different x what you have is a point on the circle on the unit circle and you can from there you can find the x and y coordinate and therefore you can actually find sine and cos of any x like that but one thing to be seen is that this value of b and value of a for any point p on the circle has to be less than one the reason being that the reason being that this point is on the circle and its and therefore since the radius is equal to one unit and both this a and b has to be less than one unit because if you see this right angle triangle here
  so for example if you see this particular point p
  so this a is obviously less than and similarly this b has to be less than this radius here if you if you project it here then this is equal to b
  so that also has to be less than the radius and the radius is one unit
  so one thing is for sure is that both have to be less than one they can be equal to one also now for example let us say that this point p
  so this is an upper bound
  so a always has to be less than one because the the largest x coordinate of any point on this circle will not exceed this point
  so the coordinate of this point here is 1 comma 0.

  so the x coordinate of any point on the circle cannot be more than one that is why a is less than one similarly the y coordinate of any point on the circle cannot be because this point is zero one
  so no y con coordinate can be above or ah in above this ah particular line let us say because we have this line here
  so no y coordinate or no point will be above this line
  so therefore b has to be less than equal to one on the other side for example if we the moment we rotate it more than ninety degrees
  so let us say we have a point here q
  so obviously as you can see the x coordinate of this point is negative and the largest negative value that the x coordinate of any point on this circle can be is when we reach let us say we rotate and reach this particular point whose coordinate is minus one comma zero
  so the x coordinate of any point has to be greater than equal to minus one similarly the y coordinate also has to be greater than equal to one
  so as you can see the range of both this sine and cosine function
  so this is the range of both the cos because sine x is b and cos x is a
  so the range of both the sine and cosine function is between minus one to plus one
  so there are some other properties that given that we have defined these functions we can discuss some properties that these functions will satisfy
  so if we go back to the previous slide where we had drawn the circle let me draw a circle for you right now
  so this was the x coordinate x axis and this is the y axis here and we had drawn this point p with x and y coordinates as a and b respectively this was x this was a and this length was b this is o let us say that this point here is is a and we had said that sine of x is equal to b and cos of x is equal to a now if you look at this right angle triangle over here o a p then from the pythagoras theorem we know that o s square
  so the o is the length of this segment o a o a square plus a p square is equal to o p square and therefore this now this o a is nothing but a
  so essentially what we are saying is that a square plus b square is equal to o p square now since this is a unit circle this o p is equal to one

so this is equal to one and a is nothing but cos x
so what we get is cos square x plus d sin x sin square x equals one
so for any x sin square x plus cos square x is always one now we already we said that sign x and cos x will lie between minus one and plus one
so when does it happen that sign x becomes zero if you again look at this circle here with points on the circle sin x is equal to the y coordinate of the points on this circle x being the angle of rotation
so sin x equal to zero basically means that we want to find out for which point does it happen that the y coordinate becomes equal to zero
so that of course if you look at this circle and all the points on the circle there are only two points where the y coordinate is zero
so one is this point here which is one zero now for this point the angle x is equal to zero
so this is one solution that sine x is zero when x equal to zero the other point where the y coordinate is zero is this point and you reach this point from this point by rotating this ray or this radius by pi radians or 180 degrees which is essentially half a revolution
so for sure we see that sign x equal to zero when either x equal to zero or x equal to pi but then we must also realize one thing that both these functions sin x and cos x their value will repeat if we increase or decrease x by multiples of two pi because two pi radians corresponds to one complete revolution
so for example suppose we consider this point p where sin for which sin x is equal to the y coordinate now we rotate this starting from o p if we move it in this direction and make one complete rotation
so so instead of this x what the angle that we are going to have is we are going to do something like this
so one complete rotation and then another x
so the angle that we are looking now is not x but x plus two pi radians right but what we realize is that the coordinates of the point correspond because after x plus two pi radians also we reach the same point p
so therefore it can be concluded that and since the we reach the same point the x and y coordinates are definitely going to be this to be the same and therefore we can conclude that sine of x and sine of x plus two pi are the same and the same thing will happen if we see x plus four pi or x plus six pi because adding two pi radians is just going one complete revolution and that is not when you go one complete revolution you do not change the we basically come at the same point on the circle
so we can write that in general sine of x is equal to sine of x plus k times two pi radians for any integer k and the same thing is true for cosine also cosine of x is equal to cosine of x plus two pi and that is also equal to cosine of x plus four pi and therefore cosine of x is also equal to cosine of x plus k times two pi for any integer k and therefore now going back to this problem where we started from which was to find out those values of x for which sign x is equal to zero in addition to x equal to zero and x equal to pi there will be lot many other solutions because since x equal to zero is a solution x equal to zero plus two pi will also be a solution and therefore four pi will also be a solution to this equation sin x equal to zero and therefore from that we can conclude that sign x will be zero equal to zero implies n is implied by x being equal to an integer multiple of pi
so you take any integer multiple of pi if you take the sign of that angle you will get sign x equal to zero
so k could be any integer
so it could be negative also

so in this class what we studied was a little bit of background of what all you had studied in your grade 10 and we then try to generalize the basic trigonometric ratios to two trigonometric functions sine of x and cos of x and we discussed some basic properties of these two functions in the next class we are going to continue with some more properties of these two functions and later on on further discuss more functions like tan of x and other function thank you you