welcome students welcome to the final lecture on relation

so finally we are going to today we will have to see some more things on functions in the previous lecture we saw some few facts about what are known as the certain facts like the properties of how ah properties of functions how do they behave on intersections and unions today let us take up the take up how the function behaves on the complement now suppose from you have a function from set x to a set y

so if a subset of x if you have a subset of x now the question is does the following hold true let us look at a simple

so ah the most thing is what we wanted is this one f of x minus a is it same as f x minus f a does the following equality hold true now before we try to understand this let us try to look at an example you consider map f or a function f from minus four the interval minus four to four to r given by f x equal to x square let us look at this function now choose a as open 0 closed 4.

now here if you notice x minus a the complement of a in x it is going to be close to minus four to close to zero and if you look at f x it is exactly this much close to zero to close to sixteen well f of a is open zero to close to sixteen you have the whole thing now if you look at f x minus f a it is exactly just singleton zero on the other hand if one tries to compute f of x minus a it is going to be close to zero at two up to close to sixteen what we have noticed here is that f x minus f e a is properly contained in f of x minus a this is what we have observed through this example now the following question is is the is it always true for all function f

so this f x minus f a contained in f of x minus a for any function f from x y in fact the answer is yes

so let us try to prove this let y belong to f x minus fe that implies y belongs to f x but y does not belong to f of a now what we have is y in f of x the image of x under f that immediately implies there exists at least one x in an element of capital x such that y is of the form f x now on the other hand y does not belong to f of a that immediately implies there does not exist any in cap an element a in a such that y is of the form f of a therefore what these two statements implies that x belongs to x but x does not belong to capital e what we have we have produced an element of capital x which is not an a whose image is y f x belongs to f of x minus a thus f x minus f a is contained in f of x minus a fine now let us look at the two things what we had f of a intersection b is contained in f of a intersection f of b and the second one f x minus f a is contained in f of x minus a what is that that is lacking for a function

so that equality hold in these cases right or what is that that we need more on f

so that equality actually holds let us look at the example that we had the example that we had is the mapping from minus four to four up to r given by f x equal to x square this was the function that we had f x equal to x square but what is that that is lacking in this if you notice this function f of two equal to f of minus two equal to four in fact what is that that we have f x equal to f of minus x that is what we have

so let us try to look at a function which does not behave in this manner

so let us begin with the definition we are looking at a function in such a manner that whenever you find an element in the code mine which is an image of some element in the domain then what we needed is that the one and only element whose image is that fixed element

so let us write it

so let f from x to y we say that fs one one or injective if f of x one equal to f of x two immediately that should imply x one equal to x two then we scale such a function as one one or injective let us do an example if we look at the

previous example f x equal to x square then f is naught one one thats what we observed it is not one to one now lets look at one more example let us look at a function from r to r given by f x equal to x cube now is this a one one function let us now show that f is one one f of x one equal to f of x two for some x one and x two then what does that imply by definition of f this will imply that x one cube equal to x two cube now taking cube roots on both sides we have x one cube the cube root of that is same as the cube root of x two cube now x one cube the cube root of that is going to give you x one and similarly the other side we have x two because the cube root of x two cube is x two therefore f is one one the natural question that one would like to ask here is in the previous example y naught one take the square root for example you have four why cannot one take the square root of four and then say that the function is one one but if you take the square root of four then you have two roots one is plus two and the other one is minus two

so you have two square roots of four and therefore the function is not one one in that case let us have one more example x equal to 1 2 3 4 and 5 and let us choose y as three four five six seven and eight now define let us look at this function f from x to y as define it as f x equal to x plus one sorry x plus two right what we have is x plus two now let us try to represent this one in a pictorial manner one two three four and five three four five six seven and eight you have these things now if you notice the function one is mapped into three two is mapped into four three is mapped into five four is mapped to six and finally five is mapped to seven right here you can notice that each element in the range that is 3 4 5 6 has got a unique pre image right

so the premade of three is exactly one and there is no other element of x which gives rise to three by of this f and similarly for four two is the only pre image and for five three is the only pre image for six four is the only premade and for seven five is the only pre image

so thus f is one one in this case in the case of in examples of this kind where the situation is easy to describe it pictorially it is always good that you draw a diagram look at that one because diagram or this kind of pictorial representation helps us in understanding whether a function is one one or not now let us look at one more concept a function f from x to y is said to be on to or surjective if co domin of f is equal to the range of f whenever co domain of f is equal to the range of f you say that such a function is onto or surjective well now let us look at let us look at the same function that we had f from minus four to four up to r given by f x equal to x square the question is is f and onto function now if you look at this one the co domain of f in this case is going to be just from 0 to 16 thus if i pick an element which is not between 0 to 16 or if i choose any element which is negative or if i choose any real number which is greater than 16 then there does not exist an x from minus four to four such that f x is going to give me that real number right

so thus if y is less than let me write it as y less than zero or y greater than 16 then we observed that there does not exist any x in minus four to four such that f x equal to y because our choice of y is such that y is less than zero or y greater than sixteen therefore f is not onto right now let us do one more example let us look at this one f from the interval zero one to r let us have it as zero one to infinity to zero comma infinity given by f x equal to one by x now is this an onto function let y belong to zero comma infinity

so this is our co domain

so let us choose an element y from the co domain now we will have to produce an element x from zero comma infinity such that f x is y

so once we have chosen this our claim is that there exists x in the open zero open infinity in this infinite interval such that f x equal to y now how to

produce this y as how to produce this x now suppose f x equal to y that is what does that means one by x is going to be my y but what we are given is this y
 so that will immediately imply that x equal to 1 by y
 so choose x as 1 by y or choose x equal to 1 by y choose x as 1 by y then f x which is 1 by x but our choice of x is 1 by y
 so this is one by one by y which is going to be therefore f is onto
 so thus we have shown that the function f x equal to 1 by x from the open interval 0 comma infinity to the open interval 0 comma infinity is an onto function now let us look at one more important concept of functions of what is known as the composition of functions suppose you have two functions f from x to y and a function g from y to z
 so given two functions f from x to y and g from y to z the composition of f and g denoted g composed with f is defined as follows
 so will use the standard rotation g composite f it is the domain of this function is going to be x and its code of mine is going to be z given by g composed with f at x equal to g of f x now let us look at some examples for composition of functions let us look at the function here from r to r given by f x equal to x square and the another function g from r to r given by g x equal to x cube let us try to compute f composed with g and g composed with x f composed g at x which by definition is f of g x which is given by f of x cube which is going to be x cube whole square which is exactly x power six on the other hand if you look at g composed with f at x which is g of f x which is same as g of x square equal to x square whole cube which is x power six you can notice in this case that f composed of g is equal to g composed with f now let us look at one more example you have from r to r given by f x equal to sin x and g from r to r given by g x equal to x square now let us try to compute f composed with g and g composed with f f composes g at x is going to be f of g of x which is f of x square which is going to be sine of x square on the other hand let's try to compute g composed with f now g composed with f r t x is going to be g of f x which is same as g of sin x which is equal to sine square x
 so what we have observed in this case is that g composed with f is not equal to f compose g
 so thus to compute compositiion the order in which we compose is very important
 so the composition may not commute thats what we have observed from this example that is g composed with f may not be equal to f composed with g let us try to look at one more example let us look at a simple example a equal to 1 2 3 4 and 5 b as 0 1 4 9 10 16 20 25 and 30 and c as 0 1 2 3 4 5 6 7 eight nine ten eleven twelve thirteen fourteen up to fifteen now let us look at the function f from a to b given by f of a equal to a square and g from b to c given by g of b as square root of b if b is a perfect square whenever it is a perfect square define it as root of b otherwise define it as b by 2 if b is not a perfect square if it is a perfect square define it as root b if it is not a perfect square define it as b by 2.
 now try to let us try to look at the composition of these two functions let us try to compute g composed with f as a function from a to c now g of let us write it as g composed with f at a equal to g of f of a which is given as g of a square but a square is always a perfect perfect square and therefore this is going to give me just square root of a well the choice of square root is only the positive square root right and therefore what you will have is square root of a square which is going to be just a thus g composed with f at a is exactly a if you look at the function g then it is it looks like a quite complicated function which is taking one values or that is which is taking you square root of b at some points and it is also taking b by 2 at some other point some other set of points it is quite complicated function but if you look at the

composition then it is going to be very simple

so sometimes composition makes things much much clearer now let us try to characterize what we wanted

so we actually started with the following f of a intersection b is contained in f of a intersection with f of b that is where we started with what we saw from those examples is that f is not one one now the question is that suppose that f is one one is it true that f of a intersection b is equal to f of a intersection f of b in fact the following statements are equivalent what are our statements first one is that f a f of a intersection b is equal to f a intersection f b e second one fs one one and third one for any two disjoint sets a and b f a and f b are disjoint the first one is f of a intersection b is equal to f of a intersection with f of b second one is f is one one third one for any two disjoint sets a and b f of a and f of b are disjoint fine now let us try to prove this statement now let us try to prove this one two implies one

so suppose that f is one one

so let us begin with what we know we know that f of a intersection b this is contained in f of a intersection with f of b

so what we will have to prove is the other way inclusion that is f of a intersection with f of b is contained in f of a intersection b

so let us proceed this way let y belong to f of a intersection with f of b that immediately implies that y belongs to f of a and y belongs to f of b now y belongs to f of a implies there exists an element a in capital a such that y is of the form f of a similarly y belongs to f of b that immediately implies there exists b in capital b such that y is of the form f of b now what we have is that we have an element a in a says that y is of the form f of a and similarly we have an element b in b such that y is of the form f of b therefore what do we have f a equal to y which is also same as fb

so what we have is f of a equal to f of b since f is one one by definition of 1 1 f a equal to f b implies a equal to b that is a belongs to a intersection b this is the one that was lacking in all those examples that we had earlier for example f x equal to x square this is the one that was lacking the one one was lacking

so that we cannot prove the we could not prove the reverse inequality therefore y which is f x belongs to f of a intersection b now let us look at the next equivalence one implies three suppose that f of a intersection b equal to f of a intersection with f of b now let a and b be disjoint subsets of x

so what we have is f of a intersection b is equal to f of a intersection f of b this holds for all subsets a and b now what we are given is that a and b are any two disjoint subsets of x that is a intersection b is m t now the way we have defined a function it is a function from a non empty set x to a non empty set y

so by convention what we always choose is that the image of an empty set is empty

so this is by convention and therefore f of a intersection b which is going to be f of m t is going to be just an empty set but by our assumption f of a intersection b is going to be f of a intersection with f of b which is going to be by our assumption is f of a intersection b which is empty that is what we have shown is that f a and f b these two sets are disjoint having said these two now lets prove the third equivalence three implies two suppose that f takes suppose that f takes disjoin sets two disjoint sets then what we will have to show is that f is one one

so let x one comma x two belong to the set x suppose that f of x one equal to f of x two

so in order to show that f is one one what we will have to show is that x one equal to x two now suppose that you have these two f of x one equal to f of x

two
 so let us assume to the contrary that x one is not equal to x 2 that is f is not 1 1 right what we have assumed is that f is not 1 1 now we will produce a contradiction how to produce a contradiction what we have is that two element distinct elements x one and x two which are x one and x two but f of x one equal to f of x two
 so x one not equal to x two that immediately says that this two single ten sets singleton x one and singleton x two are disjoint but then our assumption says that f takes disjoint sets to disjoint sets that implies that f of singleton x one is not equal to f of singleton x two that means this set is exactly f of x one this is not same as the set f of x two what we have shown is that the set containing f of x one and the set containing the singleton f of x2 these two are not one and the same that immediately implies that f of x one is not equal to f of x two which is a contradiction thus f is one one or an injective function now finally let us give a characterization of one one functions in terms of composition and onto function let f from a to b then f s one one if one only if there exists a function g from b to a such that first one g composed with f is going to be just an identity function on a and second g is one two you need an onto function g from b to a such that g composed with f should act as an identity function on a this is what we wanted now let us try to prove this let us look at the forward implication suppose that f is one one you have you are given a function that is one one now what we will have to produce is a function g from b to a
 so define well ah let us have a diagram at one end let suppose this is x this is y you have one two three and four and again you have well let us call it as one two three four and five
 so what you have is one is mapped into two two is mapped to one three is mapped to five and four is mapped to three this is the function that we have now define
 so let us keep this example as our model and then try to define what this g is going to be g from b to a as follows g of b now if you look at this example natural choice for one in y is going to be just two and similarly for two it is going to be for one for three it is four and for 5 it is 3
 so let us define it as a if b is of the form f of a
 so if you look at the elements one two three and five and y in the example that we have right there all these are going to be all these are just images of elements of x and therefore this makes sense
 so for the only thing that is left out is 4 what we need is an element of x
 so let us fix any element and then any element of x and then let us define it as arbitrarily
 so let us define it as a dash otherwise let us fix one element which is not in the range for that element let us choose an arbitrary one element and then map that b to this a dash
 so this a dash is the choice that we make it depends on one's own choice
 so now we will have to go with true proving two things one is that g is on two and the other one is that g composed with f is the identity function well lets prove the first one g composed with f at b at a is by definition g of f of a if we look at the definition of g it says the following whenever my b is of the form f of a map it to a now i have an element f of a therefore this will go to e which is exactly the identity function on a evaluated at a this is what i wanted second one continuous this g is on to let a belong to capital a i will have to produce an element b in capital y or b in capital b such that g of b is a but whenever i have an a in a f of a will be mapped to a by definition of g
 so let me choose b
 so let b choose b as f of a then g of b is going to be g of f of a which is

exactly a thus f is onto now let us try to prove the reverse part or the converse part suppose there exists an onto function g from b to a such that g composed with f is the identity function on a what i will have to show is that f is one one

so let us try to verify the definition suppose that f of a one equal to f of a two you are given that f of a one equal to f of a two what we will have to show is that a one equal to a two but once you know that f of a one equal to f of a two that will imply that g of f of a one equal to g of f of a two this is same as writing the following g composite f at a one equal to g composed with f at e two that will imply but what we know is that the composition g composed f is exactly the identity function that will immediately tell us that a 1 equal to a 2 thus f is one now a similar question that arises is that is there a similar characterization for onto functions in fact the answer is yes

so let f from a to b then fs onto if and only if there exists a function g from b to a such that first one f composed with g is the identity function on b and second g is one one

so if you have a one one function the corresponding function from g is going to be onto and if you have an onto function f from a to b then the corresponding function g from b to a is going to be one one fine let us look at the proof of this let us look at let us again ah have a similar diagram for the one part let me call this as a one two three four and five and now on the other hand let me have this one well let me have a simple set

so that things will be clear one is mapped to one two is mapped into two three is mapped to one four is mapped into two and five is also mapped into two this is the functions and what you have is an onto function

so the forward implication suppose f is onto i will have to define a function g from b to a

so for each first observe this following for each b in b let me define a set a b as all those a in a such that f of a is b here if you look at the element one then this a one will have one and three and a two will have two four and five

so fix a unique a in a b

so this a depends on the element b

so let me also write that this is a b

so that it says that this is from the set a b

so which means for each b in the code of mine i have chosen an element a from the set a now it is clear how to define g g from b to a as g of b equal to a b this choice always exists such a choice always exists because g is on 2 what will have to do is that we will have to pick one element from each of these sets a b

so once we have defined this what we will have to do is that show that first thing f composed with g is the identity on b and the second one is g is one one well lets verify one by one first one let us look at f composed g at any b which is f of g of b but g b is a b a b small a b this comes from the set capital a b capital a b consists of all elements of the set capital a which are mapped to the element b and this small a the choice is this is one of the choices one among the choices and we have made a unique choice right and from that set and therefore this is going to give me b which is exactly the identity on that the second one what we wanted is we will have to verify that g is one one suppose g of b one equal to g of p two lets apply f on that f of g of b one equal to f of g of b two that will imply f composed g at b one equal to f composes g at b two but f composes g is the identity function therefore b one equal to b two thus g is one one lets prove the converse part suppose there exists a one one function g from b to a such that f composed with g is the identity function we will have to show that g is on to let b in b i will have to produce an element a in a such

that f of a is b

 so the best choice is let me choose a as g of b then what we have is that i will have to show that f of a is b

 so therefore f of a which is going to be f of g of b which is f composed of the g at v but f composed with g is the identity function which is exactly b thus f is on

 so these are the two characterizations that we had one is the characterization of one one function in terms of an onto function and the other one is the characterization of an onto function in terms of the one one function and with this let me stop thank you all you