

welcome students in the last class we saw the notion of relations now in this class we are going to study the new class of relations called as equivalence relation in the last class when we finished we defined the notion of what is known as a symmetric relation let us begin with the same definition let us begin with the definition of a symmetric relation let A be a non empty set and let R be a relation from A to itself we say that R is symmetric if the pair (a, b) belongs to R implies the opposite pair (b, a) is also in R let us also review the same example that we last class let us look at the same example the set that we had was $\{1, 2, 3, 4, 5\}$ and R is all those (x, y) from $A \times A$ with the condition that the difference between x and y is an odd number we wrote the set R explicitly let us do it once again let us write this R again $R = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 5), (5, 4)\}$ the difference between one and two is one which is an odd number

so you have one two well one comma one the difference is zero therefore one comma one does not appear and one comma three the difference is two one comma four the difference is three therefore it appears one comma five does not appear because the difference is four now two comma one that is here because the difference is one two comma two the difference is zero two comma three the difference is one two comma four here the difference is two two comma five the difference is three next one three comma one the difference is two three comma two the difference is one three comma four and then three comma five the difference is two now let us go to four four comma one the difference is three four comma two the difference is two four comma three four comma five and then we will have five comma one the difference is four five comma two and five comma four these are the elements of R if we look at this R then it is clear one thing that can be noted is the following what is that that one can notice whenever the pair (a, b) is there then the opposite pair (b, a) is also there for example one comma two is there and similarly two comma one similarly again next one one comma four is there and you can notice that both one comma four as well as four comma one is there similarly one comma two and two comma one two comma three three comma two two comma five five comma two four comma five phi comma four

so this says that R is a symmetric relation

so therefore R is a symmetric relation let us look at one more example let $A = \{1, 2, 3, 4, 5, 6\}$ and R is all those (m, n) from $A \times A$ with the condition that n divides m right what we have is all those pairs (n, m) such that n divides m now let us write down the set R explicitly now 1 divides 1 therefore you have this $(1, 1)$ 1 divides in fact all the numbers

so we will have one is paired with all one four one five one six now going next to two you have you see that two does not divide one but two divides two but it does not divide three but it divides four it does not divide five but divides six next to three does not divide one or two but it divides itself three divides does not divide four or five but rather divides six and then for four we have only one four comma four and similarly you have four comma five and sorry ah five comma five and six comma six these are the things that you have right these are the elements now notice that you have one comma two

so what is that that you will have to note one comma two belongs to R but the opposite pair two comma one does not belong to R similarly you you have this the pair two comma six this is in R but the opposite pair six comma two is not in R

so these two things says they say that R is not a symmetric relation or is not a symmetric relation now let us do one more example let $A = \{1, 2, 3, 4, 5\}$ and let me choose $\mathcal{P}(A)$ as the power set of A this consists of

so this is the set of all subsets of A define a relation R on $\mathcal{P}(A)$ as follows two subsets A one comma A two the pair (A_1, A_2) this is in R if A_1 is

contained in a_2 for any two subsets a_1 and a_2 of A we say that a_1 is related to a_2 or the pair a_1, a_2 is in R if a_1 is contained in a_2

so one can notice the following that singleton $\{2\}$ is contained in A and not only that the other subset $\{2, 3\}$ this is also contained in A further one can notice that singleton $\{2\}$ is contained in $\{2, 3\}$ that implies that the pair $\{2\}, \{2, 3\}$ is in R right you have the set in R but on the other hand it is not true that $\{2, 3\}$ is not contained in $\{2\}$ because $\{2, 3\}$ has got two elements while $\{2\}$ has got just only one element that means that the opposite pair $\{2, 3\}, \{2\}$ this does not belong to R therefore R is not symmetric now let's do let's go to the next concept let's define one more thing let A be a non empty set and let R be a relation from A to itself we say that R is reflexive if the pair a, a belongs to R for all a in A let us look at the last example that we had the last example is our set A is $\{1, 2, 3, 4, 5\}$ and X is the power set of A we defined R as the pairs a, a' in X with the condition that a is contained in a' what is that that we know is that every set is contained in itself and therefore if b is contained in A that will imply that the pair b, b belongs to R therefore what it says b, b belongs to R therefore R is reflexive now let us go back to the one of the examples that we did again we had our A as $\{1, 2, 3, 4, 5\}$ and our R is all those x, y in X with the condition that x divides y now in this example we wrote what R is explicitly just few minutes back one can notice that one always divides one two always divides two three always divides three four divides four and similarly finally five divides five which means all this pair this whole thing this all these elements this set is contained in R that implies that R is reflexive let us look at one more example let us choose this one $\{1, 2, 3, 4\}$ and then let us say that R equal to all those n, m such that m equal to n^2 first thing that you can notice that the pair $1, 1$ is in R because one is just the square of one but the pair $2, 2$ does not belong to R which implies that R is not reflexive in the same example one can also notice the following that the pair $2, 4$ belongs to R but the opposite pair $4, 2$ does not belong to R

so this also says that R is not symmetric now let us move on to the next definition let A be a non empty set and let R be a relation from A to itself we say that R is transitive if the following holds whenever the pair a, b and b, c belongs to R that implies these two together should imply that the pair a, c is also in R we want two pairs a, b and b, c notice that the first element of the second pair is same as the second element of the first pair whenever a, b and b, c are in R that should imply that the pair a, c is in R then we say that such a relation is transitive relation now let us look at the examples that we just had the first example that we had is A equal to $\{1, 2, 3, 4, 5\}$ and then R is all those n, m and A with the condition that n divides m this is what we had now notice the following ah let us increase the set A a bit more

so that things will be clear let us increase the set notice that $2, 4$ this is in R and $4, 8$ this is in R also that $2, 8$ is also in R right in fact whenever n divides m and m divides k these two statements together implies that n divides k right therefore R is transitive now let us look at the other example that we had just few minutes back A equal to $\{1, 2, 3, 4, 5\}$ and X is the power set of A and R is all those pairs a, a' in X with the condition that a is contained in a' we saw that this is reflexive but not symmetric let us verify whether this is transitive or not

so suppose a_1 comma a_2 belongs to r and a_2 comma a_3 belongs to r now the first one a_1 comma a_2 belongs to r that tells us that a_1 is contained in a_2 the second one a_2 comma a_3 belongs to r that tells us that a_2 is contained in a_3 therefore what do we have a_1 contained in a_2 and the other hand you have a_2 contained in a_3 thus these two containments together implies us that a_1 is contained in a_3 that imply that is what do we have the pair a_1 comma a_3 belongs to r therefore r is transitive let us look at the another example same thing that we had just few minutes back let us have a as one two three four five and r is all those n comma m with the condition that the difference between n and m is an odd number now here suppose the pair n comma m is in r and m comma k this is an r now what can we say about n and k this is what is the question

so now let us look at some of the examples that we had now let us look at this one in this example we have one comma two is this true that's what we wanted n comma n in r and m comma k in r does that imply that the pair n comma k is in r this is what we wanted now we have we know that n comma one comma two is in r and two comma three this is in r because the difference between one and two is one and the difference between two and three is also one but the difference between one and three is two therefore this does not belong to r therefore r is not transitive now lets move on to the main thing that we wanted the main thing the title as we mentioned in the beginning that we are going to say about what are known as the equivalence relation a relation r from a non empty set A to itself is called an equivalence relation if one are asymmetric to r s reflexive and finally we have r s transitive whenever r is symmetric reflexive and transitive then we say that such a relation is an equivalence relation let us look at some examples let us look at a equal to one two three four five and then r as all those n comma m with the condition that n divides m what is that that we know what we did is we in fact found that r is not symmetric

so again r is reflexive third r is transitive these are the things that we noticed

so the first one r is not symmetric that says that

so this implies that r is not an equivalence relation right let us move on to this next example that we had we had A as one two three four five and then X as the power set of A all subsets of A and then we define r as pairs a one comma a two and p a cross p a with the condition that a_1 is contained in a_2 here what we saw is that r is not symmetric second well r is reflexive but r is not transitive

so the first one r is transitive right these three things together

so the first one and second one these two says that sorry the first one says that r is not an equivalence relation but now the question is does there exist a relation which is an equivalence relation yes let us do one more example let us consider let Z denote the set of all integers

so define r as follows define the relation r on Z as follows

so we say that the pair n comma m belongs to r if n is congruent to m modulo 9 right what is that n congruent to m that is the difference n minus m is divisible by nine this difference should be divisible by now let see that this relation whatever we have defined this is an equivalence relation now first thing what you will have to see is that this is symmetric suppose n comma m belongs to r that means what that is n is congruent to m modulo nine that is the difference n minus m is divisible by nine that is what does this tell us there exists an integer k such that n minus m is k times nine

so what we have is n minus m is equal to k times nine let us write down n minus m is k times nine that implies that m minus n is minus k times nine that means what m minus n this difference is divisible by nine that means m is congruent to

n modulo nine that means what is that that we have the pair m, n is an r therefore r is symmetric now second one let us verify whether r is reflexive or not

so let n belong to Z then notice that $n - n$ is 0 which i can also write it as 0 times 9

so that implies that n is congruent to n modulo nine that implies that the pair n, n is in r that is r is reflexive now let us verify whether r is transitive or not let n, m belong to r and m, r belong to capital r

so n, m the pair n, m belongs to capital r that says that that implies that $n - m$ is n is congruent to m modulo nine that implies that $n - m$ is divisible by nine that implies that there exists an integer k in Z there exists an integer k in Z such that $n - m$ is k times nine lets call this equation as one on the other hand what we have is that the pair m, r is in capital r that implies that this is equivalent to saying that m is congruent to r modulo nine that is equivalent to saying that the difference $m - r$ is divisible by nine that implies that there exists an integer p in Z such that the difference $m - r$ is of the form p times nine fine now what is that that we wanted is the pair n, r whether the difference $n - r$ is congruent to nine or not or it is divisible by nine or not what is that is what we wanted let us call the last equation as two what is that that we have $n - m$ is equal to k times nine and $m - r$ is equal to some s times nine p times nine in fact these are the two things that we had fine now with these two things let us try to calculate $n - r$ which is equal to $n - m + m - r$ equal to let me put the first two things in a single bracket and the second and third in the other bracket i am just adding and subtracting m for the first one we have $n - m$ which is k times nine plus $m - r$ which is p times nine which is equal to $k + p$ times nine now k is an integer and p is also an integer therefore $k + p$ is an integer

so let me say that note that $k + p$ is an integer therefore $n - r$ which is equal to $k + p$ times nine implies with $k + b$ and integer therefore that implies that $n - r$ is divisible by 9 that is n is congruent to r modulo 9 that implies that the pair n, r belongs to capital r

so thus we have shown that r is an symmetric reflexive and a transitive relation therefore r is an equivalence relation and now let us do one more example this example is related to geometry let A be the collection of all so all those deltas

so with the condition that delta is a two that is a triangle in r or a two dimensional triangle right what you have is a triangle in r two now let us define a relation on r r equal to all those delta one compare delta one comma delta two in a cross A with the condition that delta one is congruent to to delta it is clear from our elementary classes that first thing is that r is symmetric why is it symmetric what we know is that if delta one if triangle delta one is congruent to went to delta two that implies that delta two is congruent to to delta one therefore in fact these two are equivalent ones therefore are symmetric and similarly every triangle is congruent to itself every triangle is congruent to itself that means what r is reflexive now the third one transitivity suppose delta one comma delta two belongs to r and delta two comma delta three this pair also is an r what we will have to show is that delta 1 delta 3 belongs to r

so delta 1 comma delta 2 belongs to r that implies that delta 1 is congruent to is congruent to delta two similarly a delta two comma delta three belongs to r implies delta two is congruent to delta three right we have these two

so now what does this say that

so these two together

so let me call the first one as one and the second one has two these two together

so one and two implies that Δ_1 is congruent to Δ_3

so that means that implies that the pair Δ_1, Δ_3 belongs to R therefore R is an equivalence relation now lets do one more similar example which again comes from the geometry of the euclidean plane or the two dimensional euclidean geometry let A be the same set all those Δ such that Δ is a two dimensional triangle or a triangle in R^2 right what you have is a two dimensional triangle fine now lets define R as all those Δ_1, Δ_2 in A with the condition that Δ_1 is similar to Δ_2 the triangle Δ_1 is similar to the triangle Δ_2 now it is the same proof as what we did for the previous one one can show that R is an equivalence relation right same proof or same method as the previous example as in the previous example now let us do one more example let \mathcal{A} equal to all those A such with the condition that A is a finite set this \mathcal{A} consists of all sets which are just finite sets

so define R on \mathcal{A} as follows

so A, B come up the pair A, B belongs to R if the number of elements in A is equal to the number of elements in B

so return in set theoretic form R equal to all those pairs A, B in \mathcal{A} cross \mathcal{A} with the condition that the number of elements in A is equal to the number of elements in B now let us verify that this R is an equivalence relation first one is R symmetric

so let A, B belong to R what does that mean by definition this is same as saying that the number of elements of A is equal to number of elements of B and this is equivalent to saying that the number of elements of B is equal to number of elements of A and this implies that the pair B, A is an R and therefore this says that therefore R is symmetric now for the second one let us verify whether R is transitive to verify whether R is transitive

so let A be a finite set

so given a finite set we know that the number of elements of A is equal to the number of elements of A that implies that the pair A, A is an R therefore R is reflexive and finally the third one let the pair A, B belong to R and the pair B, C belong to R now the pair A, B belongs to R that implies that the number of elements in A is equal to the number of elements in B similarly we have that the pair B, C is in R that implies that the number of elements of B is equal to number of elements of C lets mark these as one and two therefore by one and two we have number of elements of A equal to number of elements of C

so these two together imply that R is transitive

so thus R is symmetric reflexive and transitive thus R is an equivalence relation let us do one more example again from geometry let A equal to the plane R^2 and minus the origin i am just removing the origin from the plane R^2 which is in general called as the punctured plane let us consider the punctured plane with this punctured plane let us define R as follows R equal to i will say that the pair x_1, y_1 is related to x_2, y_2 or the pair x_1, x_1, y_1 is related to x_2, y_2 belong which is an A cross A if the condition that there exists a non zero scalar λ in R such that x_1, y_1 is equal to λ times x_2, y_2 now lets verify that this R is an equivalence relation

so first one let x_1, y_1, x_2, y_2 belong to R that implies that there exists a non zero real number such that x_1, y_1 is equal to λ times x_2, y_2 now since λ is nonzero λ is invertible that means $1/\lambda$ makes sense

so therefore that implies that 1 by λ times x_1, y_1 is equal to x_2, y_2 .

so this is equivalent to saying that x_2, y_2 is equal to one by λ times x_1, y_1 that implies that this pair x_2, y_2 comma x_1, y_1 this is an r therefore r is symmetric now let us verify whether this r is reflexive or not

so let x_1, y_1 belong to A now what we know is that for any x_1, y_1 x_1, y_1 is equal to one times x_1, y_1 and since one is a non zero scalar that implies that this pair x_1, y_1 comma x_1, y_1 this is an r therefore r is reflexive now finally let's verify whether this r is transitive or not third one let the pairs x_1, y_1 comma x_2, y_2 belong to r and the pairs x_2, y_2 comma x_3, y_3 belong to r now the pair x_1, y_1 comma x_3, y_3 this belongs to r implies there exists a non zero real number such that the pair x_1, y_1 is λ times x_2, y_2 let me call this as one now similarly the pairs x_2, y_2 comma x_3, y_3 this is in r implies there exists a non zero scalar β in r such that x_2, y_2 is β times x_3, y_3 let me call this equation as two now what i am going to do is substitute two in one to get what we wanted let us do that substituting two in one we get x_1, y_1 this is λ times x_2, y_2 but then when i substitute for x_2, y_2 i get $\lambda \beta$ times x_3, y_3 but since λ and β both are non-zero real numbers that implies that this $\lambda \beta$ is a nonzero real number that implies that the pair x_1, y_1 comma x_3, y_3 this is in r therefore r is transitive thus we have shown that r is symmetric reflexive and transitive thus r is an equivalence relation

so in the next class we will do the proof of this and some more properties of this equivalence classes and then will try to define the notion of what is known as the function thank you