

today we will continue our discussion on error analysis let us recap what we have seen in error analysis whenever we take a measurement we have seen that precise measurement we may not know the exact value and therefore there is always an error in the measurement which we take and we want to quantify that error we have want to have an estimate of how much that error is so in case if we do not have the precise measurement known to us then one of the ways we do is we take many readings of the same measurement and we obtain a mean value which as we have seen is equal to sum of the individual values divided by the total number of observations based on this for each of these observations we define an absolute error and when we do this we assume that the mean value is the precise measurement so for example in reading 1 the absolute error will be a 1 the reading minus the mean the absolute value of that which means whether it is plus or minus we take it as plus so then we can define for each of these measurements we can find the absolute error and we can take the mean of all absolute errors which we call as Δa mean and that as we have seen is given by sum of all the absolute errors divided by the number of observations and once we have obtained this then we can say that the measurement which we will take it will lie between a and a minus Δa mean and a and a plus Δa mean which means Δa mean which is the mean absolute error that gives us the range in which we can expect the measurement to lie based on this we define a relative error which is equal to Δa mean divided by the mean that means we divided by the quantity error by the quantity which we are measuring and this will be a fraction and the relative error can often is expressed as a percentage and which we call as the percentage error and the symbol sometimes uses δa δa is in the small greek alphabet and that is δa a mean divided by a mean multiplied by 100 that is what the percentage would be now let us see how errors are combined and we need this for example we have a quantity which is obtained as a sum of two quantities or as a product or division of two quantities that means to find let us say we want to find the volume flow rate of certain amount of fluid which has flown so that will be equal to volume divided by t now when we measure the volume there will be some error associated with the measurement of volume similarly when we measure time there will be some error associated with measurement of time so now what we want to know is when we measure the flow rate ah using this formula when we write it we do not measure the flow rate directly we use the formula to calculate the flow rate then how much is the error in the flow rate we know the error in the volume we know the error in time from this how do we estimate the error in the flow rate and for this we realize the formulas we have typically we can divide them into two categories so let us see how errors can be combined and we how do we find an error in a quantity which is obtained as a combination of different measurements so we will look at combination of errors and here so first we look at quantities which are obtained as a sum or difference so suppose we have a quantity a which is given as $a \pm \Delta a$ here Δa is the error in a then we have a second quantity b which is given as $b \pm \Delta b$ where Δb is the error in b and we want to there is a quantity z which is equal to $a + b$ the sum of a and b and we want to find we want to find the error in z given the error in a and the error in b so then what we do is the error in z we can write it as $z \pm \Delta z$ and this will be equal to a which will be $a \pm \Delta a$ plus b which is $b \pm \Delta b$ so this if we expand this will be equal to $a + b \pm \Delta a \pm \Delta b$

delta b

so when we take this sum of errors now the error could be on either side so whenever we have a plus minus sign in the error we take it as additive we never take it as subtractive because we want to look for the maximum error so here because z is equal to a plus b we cancel this on both sides and what we get is delta z is equal to delta a plus delta b

so when we sub up two quantities the error will be summed up and that will be the error of the sum quantity similarly let us look at the subtraction and there this is where we will have to be slightly careful because suppose z is equal to a minus b then if we use the same thing as we have done before z plus minus delta z this will be equal to a plus minus delta a minus b plus minus delta b and this will be equal to a minus b plus minus delta a and once again we will get plus minus delta b and

so now if we are looking for the maximum error what we will get is this the maximum error here will be given by

so we have plus minus delta a plus minus delta b and if we look for the maximum error here then it will be clear the maximum will be whatever is the biggest number which can come from here

so that will be delta a plus delta b

so therefore even when we subtract

so if z is equal to a minus b then delta z will be equal to delta a plus delta b

so errors even though the in the original quantity the things are subtracted but when we add the errors the errors are added and this you can also look at that the error we are taking the absolute errors and when we take absolute negative becomes positive

so therefore this this is another way of seeing why these are added up next when we have a product or a quotient which means we have a quantity z which is equal to a divided by b or a multiplied by b let us first look at the product

so if z is equal to a times b then we use the same thing again z plus minus delta z is equal to a plus minus delta a times b plus minus delta b and now when we expand this this will become equal to a b and then we have plus minus delta a times b plus minus delta b times a plus minus delta a delta b and here what we do is we divide everything by z when we divide by z we divide the whole expression by z

so when we do that what we will get is 1 plus minus delta z by z this will be equal to

so let us just bring this page again and we have this here we will keep on copying from here this will be equal to now a b divided by z is a b so this will become 1 plus delta a by a plus minus delta b by b and then we have plus minus delta a delta b divided by a b

so now here once again what we will do is the plus minus will be replaced by plus because we are looking for the maximum error and the second thing which we do is that delta a delta b this product we will neglect because it is expected that the error will be small as compared to the original quantity and there is a product of two small quantities

so because it is a product of two small quantities this is neglected

so therefore what we get is the error delta z upon z is equal to delta a upon a plus delta b upon b

so this is how we obtain an error for products

so then if the product quantity divided by the original quantity is equal to some of the relative errors here and for those of you who understand logs and differentiation you will see that this formula can also be obtained by

taking logar logs on both sides and performing a differentiation but that we will leave because for now like similarly if z is equal to a divided by b then once again we will get this formula for the relative error the $\frac{\Delta z}{z}$ this will be equal to $\frac{\Delta a}{a} + \frac{\Delta b}{b}$ so once again the relative errors of the two quantities will be added now we can extend the formula suppose if z is equal to a to the power of n divided by b to the power of m now for a to the power of n we can take repeated multiples of a similarly for b to the power of m we can take repeated multiples of b and from here what you will realize is the $\frac{\Delta z}{z}$ this error will be equal to n times $\frac{\Delta a}{a}$ plus m times $\frac{\Delta b}{b}$ so the power which is there comes in as a multiplicative factor in front of the individual error now this also tells us that whichever term has the biggest power that term can be the source of biggest error so we have to be more precise when we measure that quantity so that the error associated if $\frac{\Delta a}{a}$ is less even if n is large the total contribution to the error will be less now this formula which we have $\frac{\Delta z}{z}$ is equal to $n \frac{\Delta a}{a} + m \frac{\Delta b}{b}$ will also work for the percentage error and we can because in both sides when we multiply by hundred we will get the same thing so we will say percentage error in z is equal to n times percentage error in a plus m times percentage error in b so these these are the formulas we will use when we have to find the errors of products and divisions and let us take a small example the heat generated in a wire where the current of i amperes is flowing and whose resistance is r is given by $i^2 r t$ power is equal to $i^2 r$ and when we multiplied by t that gives us the total heat so here i is the current r is the resistance t is the time and h is the heat generated so here now if we want to find out the error if it is given to us that error in measurement of i r and t are two percent three percent and one percent relative errors in measurement of i r and t are given to us and we want to find the relative error in the measurement of the heat generated so then we will go by the formula $\frac{\Delta h}{h}$ is equal to now this i^2 square so it will be two times $\frac{\Delta i}{i}$ plus $\frac{\Delta r}{r}$ plus $\frac{\Delta t}{t}$ and we express each of them as a percentage so because they are given as percentage so what we will have is $\frac{\Delta h}{h}$ in terms of into hundred and that is what will give it this in percentage will be two $\frac{\Delta i}{i}$ into hundred percent plus $\frac{\Delta r}{r}$ into hundred percent plus $\frac{\Delta t}{t}$ into hundred percent now these are given to us so this will be equal to two multiplied by two percent plus three percent plus one percent so this total will be equal to eight percent so that means the error if we put in $\frac{\Delta h}{h}$ this will be equal to in percentage this will be equal to plus minus eight percent so and as if it is required that this eight percent can be covered converted back into the units of Δh if we know the value of h and Δh we will see a similar example towards the end of this topic now having seen this concept of errors what we have seen is how do we find errors in quantities which are added and errors in quantities which are either multiplied or divided next we talk about significant figures in a measurement whenever any measurement is reported then in the measurement we have the uncertainty in the measurement is in the last digit the last digit could be either \pm maybe less or bigger by an amount of unity and the this is what we account for all this through

significant figures for example when we say the time period of a particular pendulum is 1.62 seconds then here we know one and six are reliable digits whereas when we say two this two could be the one where there could be a possibility or uncertainty is in the digit two it could be either one or three or a fraction somewhere lying between one and three so this is where the uncertainty comes

so now when we talk of 1.62 seconds we say this measurement 1.62 seconds we say this is correct up to three significant digits

so in a digit which a measurement like one point six two this is has got three significant digits we take another measurement take an example suppose we have 20 is 287.5 centimeters

so this means we are taking our measurement with a ruler which has one millimeter graduation we are going up to one millimeter so here we have four significant digits and always the uncertainty is in the last digit now the significant digits indicate the precision of an instrument which depends on the least count now what we should also realize is that the choice of different units should not affect the number of significant digits and the reason for this is that the least count of the instrument will not change whether we change the units from centimeters to millimeters it will be the same and

so for example when i have a measurement of 2.308 centimeters this has got four significant digits now if i look at it this measurement in millimeters it will be twenty three point zero eight millimeters and once again this will have four significant digits if i put this look at this in terms of meters this will be equal to zero point zero two three zero eight meters and number of significant digits in all of these should stay to be four and therefore keeping this in mind we have certain rules for determining the number of significant digits the first rule is that all non-zero digits are significant so wherever there is a non-zero digit when we see a measurement that has to be counted as a significant digit the second rule is the zeros which come between two non zero digits are significant irrespective of the decimal point and for example in this example above here when i look at 23.08 there is a 0 which comes just after that decimal place but then still this 0 will be counted when we count in significant digits because it is surrounded by 2 non-significant digits

so when we write 23.08 we will talk of this as four significant digits now the third rule these were sort of simple rules now we have to be a bit careful if a number is less than one that means the number will be something like if we express it in a decimal it will be zero point something now there if that is

so then zeros to the right of the decimal point but to the left of first non-zero digit are not significant and let me give you an example of this for example when we say 0.00238 then we observe that we are starting with the number uh the decibel the full part here is zero this number is less than one then there are two zeros after the decimal point these will not be significant

so in this number we will have three significant digits

so this was the third rule now the fourth rule is if there is a full number that means there is no decimal point then in this number if it ends with zeros the trailing zeros are usually not significant unless if the measurement has been taken up to that accuracy which means suppose if we have a measurement of one two three meters this is got three significant digits we convert this into centimeters

so the same measurement will become one two three zero zero centimeters and

here the last two digits these two zeros they will not be counted in the significant digits and the number of significant digits will be just three as before now the final rule which we have here is if there is a number with decimal point let us say for example we write a number three point five zero zero we write it like this now here because the two zeros have been included after the decimal digit that means the measurements were accurate up till this unit

so therefore in this case the number of significant digits is equal to four so the zeros the trailing zeros in numbers with decimal points are significant and sometimes to get rid of this confusion the number is reported in a scientific notation we report the number in scientific notation which means the number is written in powers of ten

so any number is expressed as $a \times 10^b$ where a is between one and ten

so a will be a number between one and ten and b is the exponent and b could be positive or negative if the number is less than 1 then b will be negative otherwise it will be positive

so b is an exponent and then how whatever is the number of significant digits in a will be the number of significant digits in that number

so this way then the ambiguity is resolved then we also define something called if we want to work out what we call as the order of magnitude where we want to have an estimate of the measurement not the precise value but how much it is for example when we have something like 1 meter or 2 meters length we call them as the order of magnitude of 1 meter but if something is 100 meters or 110 meters then we will say the order of magnitude is 100 meters so the order of magnitude how we write is again if we express then express the number in scientific notation then we will write it as $a \times 10^b$ where a lies between one and five then we say the order of magnitude is equal to 10^b or rather we call ignorant square last partner

so next we talk of order of magnitude now if a quantity is written as in the scientific notation as $a \times 10^b$ then if a lies between 1 and 5 then we round it off to one and then we say the quantity we make an estimate of this as 10^b and b is called the order of magnitude of the quantity and if a lies between 5 and 10 then we round up round off to the next digit which means we call it as 10^{b+1} and in this case $b+1$ is called the order of magnitude order of magnitudes are normally used to make rough estimates of the quantities and they not talk of any will not be very exact when we talk of order of magnitude but this is to give an idea of how much is the sense of measurement for example if something is 1.28×10^7 meters then we say the order of magnitude of this length is 10^7 meters and similarly if i talk of the diameter of hydrogen atom then we observe the diameter of the hydrogen atom is 1.06×10^{-10} meters and then we say its order of magnitude is 10^{-10} meters

so this is how we work out now in certain quantities what we find is that there are in some formulas there are some constants for example let us look at a very simple formula diameter is equal to two multiplied by radius now in this particular formula two is a number which is exact and so this has got infinite significant digits

so we do not that means we are assuming or not in this case the number is exact

so there is no error in this number similarly for example when we calculate

the circumference we have $2\pi r$ and the pi number pi can be calculated to as many significant digits as we like we can read the formula so therefore when constants like these come in formulas we do not take any error associated with them they are assumed to be exact now we will like to see another important aspect of these things and which is the rules for operations when we have quantities with errors in them and the basic rule which we have is that the final result cannot be more accurate than the original measured values

so when we calculate a quantity then we will look at the least accurate value from these measurements and whatever is the order of magnitude of the least accurate measurement that will be taken as the order of magnitude for the final quantity

so for example let's see we have density is equal to mass upon volume and let us say mass of a quantity is given to us as 4.237 grams and the volume is given as 2.51 centimeter cube and we have to calculate the density of this now today this is an era of calculators if i am if i ask anyone to do this you will take a calculator you will punch these two numbers and when you calculate the density using these two numbers the calculator will give you an answer something like one point six eight eight i am copying this zero four seven eight zero eight seven six depending on how many digits are there in your calculator now you can understand when i write an answer like this this gives a sense that the quantity is accurate up till this last digit which is point zero zero zero zero zero zero up till whatever the number here is whereas our original measurements are correct in the first case up till 4 significant digits or 0.001 grams and in volume up till 0.01 centimeter cube or 3 significant digits

so then using expressing my answer in terms of these 11 numbers after the decimal is an absurd thing to do and is wrong

so therefore how do we work out when we have operations like this then how should we calculate up till which digit should we write the final answer is the question and let us try to understand this very well because this is something we have to be very careful about

so let us see how many digits should we carry around when we have multiplication or division and then we will see for addition and subtraction now for multiplication and division we the rule is that the final result should retain as many significant figures as are there in the original number with least significant digits what this means is you are multiplying or dividing or the formula contains all this of quantities or three or four quantities first quantity has five significant digits second has six third has three and fourth has one

so then the final result which you have should contain as many significant figures as the one with least significant digits

so in this case which i said if we had one significant digit in these four quantities then final answer has to be with one significant digit and let us go back to our previous example where we said that the mass was 4.237 grams and the volume was 2.51 centimeter cube and we wanted to find the density so we see in this quantity in mass there are four significant digits in volume they are three significant digits so finally in density we will carry only three significant digits

so when we look at the answer which we had the answer which we had for the density when we use the calculator we got an answer of one point six eight eight such a number now we have to write this answer only up to three significant digits

so that means we will go up till one point six eight eight now the digit

after eight is greater than five that means we will have to round up our answer and

so if the answer has to be expressed in terms of three significant digits so the density will be given as 1.69 grams per centimeter cube i will talk a bit about this rounding in a after i talk about addition and subtraction i will give you the rules of rounding also but secondly let us take the case of addition or subtraction now it is very clear in addition or subtraction when we are adding two quantities or subtracting two quantities then these two quantities have to have the same dimensions they cannot be two different dimensional quantities for example we can add two lengths or we can add two masses but i cannot add a mass to a length

so now here what we have to go is how much is the error in each of these quantities and whichever quantity if i am adding up two quantities whichever has the most error that is the error which i will have to take so here what we do is we retain in the final result when we add or subtract we retain as many decimal places as are there in the number with least decimal places and it becomes clear with an example suppose i am adding up 3 masses there is a mass 436.32 grams there is another mass 227.2 grams and then there is 0.301 grams if i have to add up these 3 masses then if we total them up this comes out to be 663.821 grams now we look at each of these individual masses we find the second mass has got one is got only up to one decimal place that means we will finally have to write our answer correct up to the first decimal place

so this answer should be reported as 663.8 grams because in this second digit it could be 0.21 or 0.22 or 0.23

so adding something about which we are not sure will not make any sense so that is why we have to do it like this now here implicitly i have used the concept of rounding off which you must be familiar with but let us just formally see this also and rounding off means these are very obvious rules example let us say we have a digit two point seven four six and we want to write it up to three significant figures you calculate a number you get this and you want to now write it up to three significant figures three significant figures means we have to go up till this digit

so now your common sense tells you you can either put it as two point seven four or 2.75 but because this digit is bigger than half you will round it up as 2.75 similarly if you have a number 2.743 and once again you want to round it up to three significant figures then once your you will realize that up to three significant figures the rounded up number will be written as 2.74

so now the rules if we formalize them we will get the rules as if the the preceding digit is raised by one if the digit to be dropped is bigger than 5 and this is left unchanged if the digit to be dropped is less than five so we look at the digit which has to be dropped and we see if the digit to be dropped is bigger than five we raise the preceding digit by 1 and if it is less than 5 we leave it unchanged but then the question comes what happens if the digit is if your number is something like point seven four five and this you have to write up to three significant digits so if the digit to be dropped is five then there is an ambiguity and here the convention which we use is that if the digit to be dropped is 5 then we look at the preceding digit if the preceding digit is even then we drop the five

so for example when two point seven four five because the digit to be dropped is five we look at the digit before this it is four so this is an even digit

so we will drop the five

so this digit will become two point seven four and if the preceding digit is odd then we add one to the preceding digit for example if we are talking of two point seven three five up to three significant digits we look at this digit five and the digit preceding to this is three which is odd so therefore this would be rounded up as two point seven four but there is one more thing we keep in mind if we have to round up a number like 2.7351 up to 3 significant digits which means we have to retain the first three digits and the digits to be dropped are five one

so in this case because what we are dropping is bigger than five because there is a 1 following that

so this will be in that case written as 2.74 and this would even happen if we have something like 2.7451 and if we have to keep this up to 3 significant digits

so that means we are looking at 2.74 but now the part which we are dropping is bigger than half because five is followed by a one

so in this case this will be written as two point seven five now let us take one example here which will help us to understand all these things let us say suppose length of a rectangle is measured as 16.2 centimeters and the breadth is measured as 10.1 centimeters and this is using a meter scale and we want to find out the area now if we were to use a calculator we will work out a number we will multiply these two 16.2 into 10.1 and our answer will come out to be something like i think one sixty three point six two centimeter square

so we but let us look at the concept of significant digits and errors and so we will let see how to write this correctly

so if i the length since the last digit here is 0.2 centimeter the last significant digit is there

so the length we will write it as 16.2 plus minus 0.1 centimeters and similarly the breadth will be written as 10.1 plus minus 0.1 centimeters

so now when we multiply the two we will get the area and area will be equal to sixteen point two plus minus zero point one into ten point one plus minus zero point one centimeter square

so let us see how to work this out

so we can directly see this if l is equal to 16.2 ± 0.1 then Δl is 0.1 and the relative error in length which is $\Delta l / l$ this will be equal to $0.1 / 16.2$ and this if i express it as a percentage this will be equal to 0.6 percent similarly when i look at breadth $\Delta b / b$ this will be equal to zero point one divided by ten point one into hundred percentage so this is equal to one percent and area is equal to $l \times b$ and if i write the error in area $\Delta a / a$ is equal to $\Delta l / l + \Delta b / b$ upon b and i can also put multiply both sides by 100 that will give me the percentage

so what i will get is $\Delta a / a$ as a percent this will be equal to point 0.6 percent plus one percent

so this is equal to one point six percent

so Δa the error in the area this will be equal to 1.6 by 100 multiplied by the value of a and a is equal to 16.2 into 10.1 so therefore what i get is area is equal to 163.62 centimeter square plus minus 1.6 percent now this is where we will have to work out the total number of significant digits in the original quantities for each of them for length as well as breadth word 3

so the final answer when we express we have to express it in the same number of significant digits and this is the mistake which a lot of people make

so therefore here now when i express it to 3 significant digits this will become 164 centimeter square that means the last two the decimal places will go away because we keep only the significant digits and then to this i have to add one point six percent now 1.6 percent of 163.62 this becomes equal to 2.6 centimeter square but i have to this 2.6 which i get i have to keep it i'm adding it up to here to 164.

so i have to keep it as the same number of decimal places as 164.

so therefore this 2.6 will be rounded up to 3 centimeter square and so the final expression for the area will be 164 centimeter square plus minus 3 centimeter square

so this is how one expresses the a quantity which is obtained by a product now one thing which will realize is if we look at relative error relative error this depends not only on the number of significant digits but also on the number being measured

so for example when i look at a mass being measured which is 1.02 grams and i measure it up to an accuracy of 0.01 grams then the relative error here is 0.01 divided by 1.02 into 100 and this will be equal to 1 percent whereas if the same relative error is there if there is a mass of almost 10 grams let us say 9.89 grams which has also measured up to an accuracy of 0.01 grams in this case the relative error you will realize will be much less this will be equal to 0.01 divided by 9.89 and this when i express it as a percent this turns out to be equal to 0.1 percent

so for the same least count if the original mass was more the relative error in that mass is much less as compared to a lighter body now this one another rule which we use if you have multiple step calculations then what we do is so this we should keep in mind when we have multiple step calculations then in the intermediate steps we retain one extra digits to take care of errors which could creep in because of multiplication divisions etcetera we keep one extra digit and in the final answer we will follow the rules so when we calculate write the final answer we keep the number of significant digits as per the rules which we have defined but in the intermediate steps we retain one extra digit thank you you