The Straight Line. (2D)

* The general equation of the first degree, in x, y, i.e. $ax + by + c = 0$ represents a straight line.

* To completely determine the equation of a straight line, we require two conditions.

* Slope of a line: If inclination is θ, then $\tan\theta$ is called the slope or gradient of the line. If (x_1, y_1) & (x_2, y_2) are two points on a line, then the slope m is $(y_2 - y_1)/(x_2 - x_1)$. $(x_1 \neq x_2)$

● If three points A, B, C are collinear, then $m(AB) = m(BC) = m(CA)$.

* Angle between two lines: The acute angle θ between the lines having slopes m_1 & m_2 is given by

$$\theta = \tan^{-1}\left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$

● If two lines are parallel, $m_1 = m_2$ & if perpendicular $m_1 m_2 = -1$.

* Lines parallel to co-ordinate axes:
i) to y axis : $x = a$ (equ^n of y axis $x = 0$)

ii) to x axis: $y = b$ (equ^n of x axis $y = 0$).

* Intercepts of a line on axes: x intercept a, y intercept b

quad I -	$+a, +b$	quad III - $-a, -b$
quad II -	$-a, +b$	quad IV - $+a, -b$

* Different forms of equⁿ of st. line:

1. Slope-intercept form: Equⁿ of st. line whose slope is m & which cuts an intercept c on y-axes, is

$$y = mx + c.$$

2. Point-Slope form of line: Equⁿ of line that passes through (x_1, y_1) & has slope m is $y - y_1 = m(x - x_1)$.

3. Two-point form: Equⁿ of line passing through two given points (x_1, y_1) & (x_2, y_2) is $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

or $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

4. The Intercept form: Equⁿ of line which cuts off intercepts of length of a & b on x axis & y axis respectively, is $\dfrac{x}{a} + \dfrac{y}{b} = 1.$ or $\begin{vmatrix} a & o & 1 \\ x & y & 1 \\ o & b & 1 \end{vmatrix} = 0.$

5. Normal/Perpendicular form: Equⁿ of line upon which the length of perpendicular from origin is p & normal makes an angle α with the direction of x-axes is

$$x \cos α + y \sin α = p.$$ [p is always +ve]

or $\begin{vmatrix} p\sec α & o & 1 \\ x & y & 1 \\ o & p\csc α & 1 \end{vmatrix} = 0.$

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6. **Symmetric/Parametric Form:** Equⁿ of st. line passing through (x_1, y_1) & (x_2, y_2) & making an angle θ with the +ve direction of x-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r.$$

where r is the directed distance between the points (x, y) & (x_1, y_1).

- $\left.\begin{array}{l} x = x_1 + r\cos\theta \\ y = y_1 + r\sin\theta \end{array}\right\}$ parametric equⁿ's of st. line

* **Reduction of general equation to standard form:**

1. **Slope-intercept form:** $y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$.

$$[B \neq 0]$$

$$\text{Slope, } m = -\frac{A}{B}.$$

$$y\text{-intercept, } c = -\frac{C}{B}.$$

- **cor.** Angle between $A_1 x + B_1 y + C_1 = 0$ & $A_2 x + B_2 y + C_2 = 0$.

$$\tan\theta = \left| \frac{A_1 B_2 - A_2 B_1}{A_1 A_2 + B_1 B_2} \right|.$$

If lines are parallel $\frac{A_1}{A_2} = \frac{B_1}{B_2}$.

If perpendicular $A_1 A_2 + B_1 B_2 = 0$.

- If two lines are coincident $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

2. **Intercept form:** $\dfrac{x}{-C/A} + \dfrac{y}{-C/B} = 1.$ $[A, B \neq 0]$.

$\quad\quad x\text{ intercept} = -C/A.$

$\quad\quad y\text{ intercept} = -C/B.$

3. **Normal form:** $\left(-\dfrac{A}{\sqrt{A^2+B^2}}\right)x + \left(-\dfrac{B}{\sqrt{A^2+B^2}}\right)y = \dfrac{C}{\sqrt{A^2+B^2}}.$
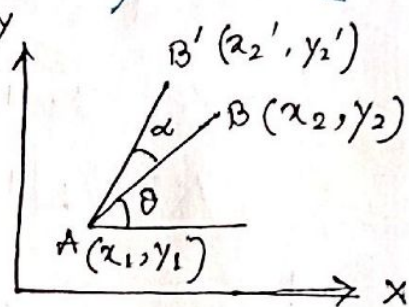
* New Co-ordinates when rotated by an angle α:

$$\dfrac{x_2 - x_1}{\cos\theta} = \dfrac{y_2 - y_1}{\sin\theta} = r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$x_2 = x_1 + r\cos\theta$

$y_2 = y_1 + r\sin\theta.$

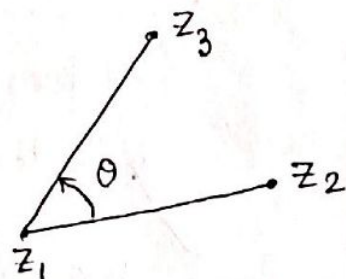$x_2' = x_1 + r\cos(\theta + \alpha)$; $y_2' = y_1 + r\sin(\theta + \alpha).$

* Complex number as a rotating arrow in Argand Plane:

$$Z_3 - Z_1 = (Z_2 - Z_1)e^{i\theta}.$$

$\Rightarrow \dfrac{Z_3 - Z_1}{Z_2 - Z_1} = e^{i\theta}.$

* **Position of a point relative to a line:**
The point (x_1, y_1) is on one side or the other side or the line $ax + by + c = 0$ $(b > 0)$ according as $ax_1 + by_1 + c > 0$ or < 0.

* **Position of two points relative to a given line:** The points $P(x_1, y_1)$ & $Q(x_2, y_2)$ lie on the same or opposite sides of the line

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$ax + by + c = 0$ according as,

$$\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0 \text{ or } < 0.$$

- **Cor.** A point (α, β) will lie on origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have same sign.

* **Equations of lines parallel and perpendicular to a given line:** 1. Equⁿ of line parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is some constant. 2. Equⁿ of line perpendicular to $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is some constant. **Cor.** Equⁿ of line parallel to $ax + by + c = 0$ & passing through (x_1, y_1) is $a(x - x_1) + b(y - y_1) = 0$. Similarly, for perpendicular line $b(x - x_1) - a(y - y_1) = 0$

* **Distance of a point from a line:** Length of perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$ is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

* **Distance between two parallel lines:** Perpendicular distance between the lines $ax + by + c = 0$ & $ax_1 + by_1 + c_1 = 0$ is $\dfrac{\lambda}{\sqrt{a^2 + b^2}}$.

i) $\lambda = |c_1 - c|$, if both lines on same side of origin.

ii) $\lambda = |c_1| + |c|$, if they are on opposite sides, of origin.

* **Area of Parallelogram:** Area of parallelogram $ABCD$ whose sides AB, BC, CD, DA are represented by $a_i x + b_i y + c_i = 0$.

$$\frac{P_1 P_2}{\sin \theta} \quad \text{or} \quad \left| \frac{|c_1 - d_1||c_2 - d_2|}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \right|$$

$$a_1 x + b_1 y + c_1 = 0$$
$$a_2 x + b_2 y + c_2 = 0$$
$$a_1 x + b_1 y + d_1 = 0$$
$$a_2 x + b_2 y + d_2 = 0$$

where, P_1 & P_2 are the distances between parallel sides and θ is the angle between two adjacent sides.

● cor. If $P_1 = P_2$ then the area of rhombus,

$$P_1^2 / \sin \theta \quad = \quad \frac{(c_1 - d_1)^2}{|a_1 b_2 - a_2 b_1| \sqrt{\dfrac{a_1^2 + b_1^2}{a_2^2 + b_2^2}}}$$

cor. If d_1 & d_2 are the lengths of two perpendicular diagonals of a rhombus then area of rhombus $= \frac{1}{2} d_1 d_2$.

cor. Area of parallelogram whose sides are $y = mx + a$, $y = mx + b$, $y = nx + c$ & $y = nx + d$ is $\left| \dfrac{(a-b)(c-d)}{(m-n)} \right|$.

* **Concurrent lines:** Three lines $a_i x + b_i y + c_i = 0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

or,

Condition for the lines $P = 0$, $Q = 0$, $Z = 0$ to be concurrent is that three constants l, m, n (not all zeroes at the same time) can be obtained such that $lP + mQ + nZ = 0$.

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* Family of lines: Any line through the point of intersection of the lines

$$a_1 x + b_1 y + c_1 = 0 \quad \& \quad a_2 x + b_2 y + c_2 = 0 \quad \text{can be}$$

represented by

$$(a_1 x + b_1 y + c_1) + \lambda(a_2 x + b_2 y + c_2) = 0. \quad \text{where } \lambda$$

is a parameter which depends on the other property of line.

* The equations of the straight lines which pass through a given point (x_1, y_1) & make an angle α with the given straight line $y = mx + c$ are

$$y - y_1 = \tan(\theta \pm \alpha)(x - x_1) \quad [m = \tan\theta].$$

* A line equally inclined with two lines:
If two lines with slopes $m_1 \& m_2$ be equally inclined to a line with slope m, then

$$\frac{m_1 - m}{1 + mm_1} = - \frac{m_2 - m}{1 + mm_2}.$$

* Equation of the bisectors: Equ^n of the bisectors of the angles between the lines $a_1 x + b_1 y + c_1 = 0$ & $a_2 x + b_2 y + c_2 = 0$ are given by,

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Equ^n of bisector containing origin $\dfrac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \dfrac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}$

Equ^n of n not containing origin $\dfrac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\dfrac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}$

* **Equation of bisector of the angle between the two lines containing points (h, k) will be,**

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

according as $a_1 h + b_1 k + c_1$ & $a_2 h + b_2 k + c_2$ are of the same sign or opposite sign.

* **Distinguishing the acute & obtuse angle bisectors :** Equⁿ of bisector

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

Condition	Acute angle bisector	Obtuse angle bisector
$a_1 a_2 + b_1 b_2 > 0$	$-$	$+$
$a_2 a_1 + b_1 b_2 < 0$	$+$	$-$

* **foot of perpendicular drawn from the point (x_1, y_1) to the line $ax + by + c = 0$:**
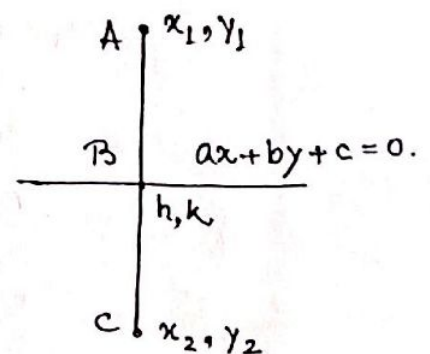
$$x = \frac{ac - b(bx_1 - ay_1)}{b^2 - a^2}$$

$$y = \frac{bc - a(bx_1 - ay_1)}{b^2 - a^2}.$$

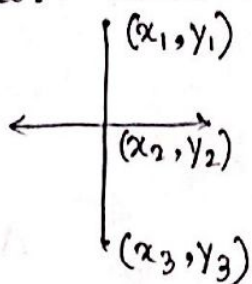* **Image or reflection of a point (x_1, y_1) about a line mirror :**

If $B : h, k$ then

$$h = \frac{x_1 + x_2}{2}, \quad k = \frac{y_1 + y_2}{2}.$$

$A \, \bullet \, x_1, y_1$

B $ax + by + c = 0$.

h, k

$c \, \downarrow \, x_2, y_2$

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* **Short-cut:**

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{ax_1 + by_1 + c}{(a^2 + b^2)}$$

$$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

* **Position of a point which lies inside a triangle:** First draw the exact diagram of the problem. If the point $P(x_1, y_1)$ move on the line $y = ax + b$ for all x_1, y_1, then $P \equiv (x_1, ax_1 + b)$ & a portion DE of the line $y = ax + b$ (excluding D & E) lies within the triangle.

Now, $y = ax + b$ cuts any two sides out of three sides, then find co-ordinates of D & E.

$$D \equiv (\alpha, \beta) \quad , \quad E \equiv (\gamma, \partial)$$

then $\qquad \alpha < x_1 < \gamma$ & $\beta < ax_1 + b < \partial$.

* **If** $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ & $C \equiv (x_3, y_3)$ are the vertices of $\triangle ABC$, then $\angle A$ is acute or obtuse according as,

$$(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3) > 0 \quad \text{or} \quad < 0.$$

for $\angle B$,

$$(x_2 - x_3)(x_2 - x_1) + (y_2 - y_3)(y_2 - y_1) > 0 \quad \text{or} \quad < 0.$$

for $\angle C$,

$$(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) > 0 \quad \text{or} \quad < 0.$$

* If the origin lies in the acute or obtuse angle between the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ according as.

$$(a_1a_2 + b_1b_2)\, c_1c_2 < 0 \quad \text{or} \quad > 0.$$

* If (x_i, y_i) are the vertices of $\triangle ABC$ then equations of right bisectors of side BC, CA, AB are

$$y(y_2 - y_3) + x(x_2 - x_3) = \left(\frac{x_2^2 - x_3^2}{2}\right) + \left(\frac{y_2^2 - y_3^2}{2}\right).$$

$$y(y_3 - y_1) + x(x_3 - x_1) = \left(\frac{x_3^2 - x_1^2}{2}\right) + \left(\frac{y_3^2 - y_1^2}{2}\right).$$

$$y(y_1 - y_2) + x(x_1 - x_2) = \left(\frac{x_1^2 - x_2^2}{2}\right) + \left(\frac{y_1^2 - y_2^2}{2}\right).$$

○ equn's of medians AD, BE, CF are

$$y(x_2 + x_3 - 2x_1) - x(y_2 + y_3 - 2y_1) = y_1(x_2 + x_3) - x_1(y_2 + y_3).$$

$$y(x_3 + x_1 - 2x_2) - x(y_3 + y_1 - 2y_2) = y_2(x_3 + x_1) - x_2(y_3 + y_1).$$

$$y(x_1 + x_2 - 2x_3) - x(y_1 + y_2 - 2y_3) = y_3(x_1 + x_2) - x_3(y_1 + y_2).$$

○ equn's of altitudes AL, BM, CN are

$$y(y_2 - y_3) + x(x_2 - x_3) = y_1(y_2 - y_3) + x_1(x_2 - x_3).$$

$$y(y_3 - y_1) + x(x_3 - x_1) = y_2(y_3 - y_1) + x_2(x_3 - x_1)$$

$$y(y_1 - y_2) + x(x_1 - x_2) = y_3(x_1 - x_2) + x_3(x_1 - x_2).$$

* If of $\triangle ABC$, BC: $a_1x + b_1y + c_1 = 0$, CA: $a_2x + b_2y + c_2 = 0$, AB: $a_3x + b_3y + c_3 = 0$, then

$$|BC| : |CA| : |AB| =$$

$$\sqrt{a_1^2 + b_1^2}\; \left|\begin{matrix} a_2 & b_2 \\ a_3 & b_3 \end{matrix}\right| : \sqrt{a_2^2 + b_2^2}\; \left|\begin{matrix} a_3 & b_3 \\ a_1 & b_1 \end{matrix}\right| : \sqrt{a_3^2 + b_3^2}\; \left|\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}\right|$$

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* Area of Triangle having corner points (x_1, y_1), (x_2, y_2), (x_3, y_3)

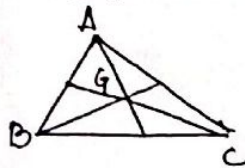$$= \frac{1}{2}\left[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

* In case of polygon with vertices (x_i, y_i), then

$$area = \frac{1}{2}\left|(x_1 y_2 - y_1 x_2) + (x_2 y_3 - y_2 x_3) + \cdots + (x_{n-1} y_n - y_{n-1} x_n) + (x_n y_1 - y_n x_1)\right|.$$

* Centres connected with a triangle:

$A \equiv (x_1, y_1)$
$B \equiv (x_2, y_2)$
$C \equiv (x_3, y_3)$
$BC = a, \quad CA = b$
$AB = c$

1. Centroid: Point of concurrency of the medians.

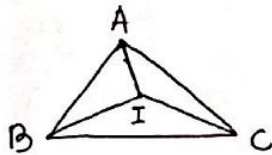$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

2. Orthocentre: Point of concurrency of the altitudes.

$$H \equiv \left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right).$$
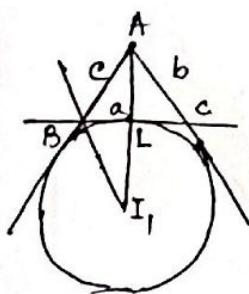
(NB – $\triangle DEF$ is orthic or pedal triangle).

3. Incentre: Point of concurrency of the internal bisectors of angles.

$$I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

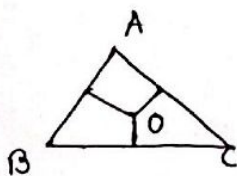**4. Excentre:** Point of concurrency of two external bisectors of angle.

$$\frac{BL}{LC} = \frac{c}{b} \qquad \frac{AI_1}{I_1 L} = -\frac{b+c}{a}$$

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right).$$

Similarly, I_2, I_3.

5. Circumcentre: Point of concurrency of perpendicular bisectors of sides.

$$O \equiv \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$