$$\frac{x^2}{\left(\frac{247}{7}\right)} + \frac{y^2}{\frac{247}{15}} = 1 \text{, i.e., } 7x^2 + 15y^2 = 247.$$

EXERCISE 11.3

In each of the Exercises 1 to 9, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

1. $\frac{x^2}{36} + \frac{y^2}{16} = 1$ **2.** $\frac{x^2}{4} + \frac{y^2}{25} = 1$ **3.** $\frac{x^2}{16} + \frac{y^2}{9} = 1$

4. $\frac{x^2}{25} + \frac{y^2}{100} = 1$ **5.** $\frac{x^2}{49} + \frac{y^2}{36} = 1$ **6.** $\frac{x^2}{100} + \frac{y^2}{400} = 1$

7. $36x^2 + 4y^2 = 144$ **8.** $16x^2 + y^2 = 16$ **9.** $4x^2 + 9y^2 = 36$

In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions:

10. Vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$

11. Vertices $(0, \pm 13)$, foci $(0, \pm 5)$

12. Vertices $(\pm 6, 0)$, foci $(\pm 4, 0)$

13. Ends of major axis $(\pm 3, 0)$, ends of minor axis $(0, \pm 2)$

14. Ends of major axis $(0, \pm\sqrt{5})$, ends of minor axis $(\pm 1, 0)$

15. Length of major axis 26, foci $(\pm 5, 0)$

16. Length of minor axis 16, foci $(0, \pm 6)$.

17. Foci $(\pm 3, 0)$, $a = 4$

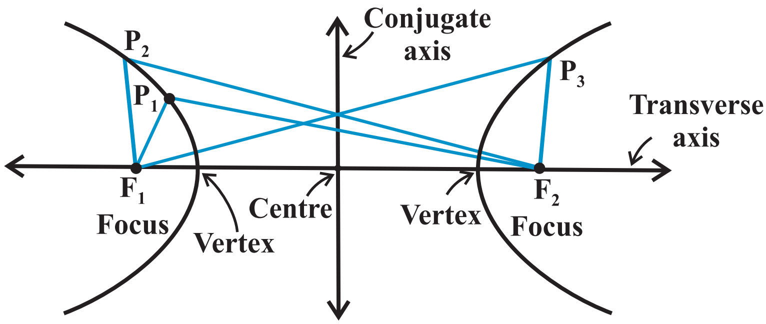
18. $b = 3$, $c = 4$, centre at the origin; foci on the x axis.

19. Centre at (0,0), major axis on the y-axis and passes through the points (3, 2) and (1,6).

20. Major axis on the x-axis and passes through the points (4,3) and (6,2).

11.6 Hyperbola

Definition 7 A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.



$$P_1F_2 - P_1F_1 = P_2F_2 - P_2F_1 = P_3F_1 - P_3F_2$$

Fig 11.29

The term "*difference*" that is used in the definition means the distance to the farther point minus the distance to the closer point. The two fixed points are called the foci of the hyperbola. The mid-point of the line segment joining the foci is called the *centre of the hyperbola*. The line through the foci is called the *transverse axis* and the line through the centre and perpendicular to the transverse axis is called the *conjugate axis*. The points at which the hyperbola intersects the transverse axis are called the *vertices of the hyperbola* (Fig 11.29).

We denote the distance between the two foci by $2c$, the distance between two vertices (the length of the transverse axis) by $2a$ and we define the quantity b as

$$b = \sqrt{c^2 - a^2}$$

Also $2b$ is the length of the conjugate axis (Fig 11.30).

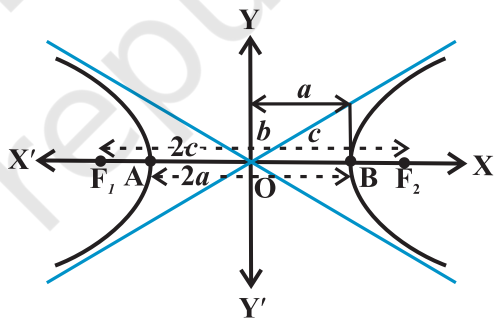


Fig 11.30

To find the constant $P_1F_2 - P_1F_1$:

By taking the point P at A and B in the Fig 11.30, we have

$BF_1 - BF_2 = AF_2 - AF_1$ (by the definition of the hyperbola)

$BA + AF_1 - BF_2 = AB + BF_2 - AF_1$

i.e., $AF_1 = BF_2$

So that, $BF_1 - BF_2 = BA + AF_1 - BF_2 = BA = 2a$

11.6.1 *Eccentricity*

Definition 8 Just like an ellipse, the ratio $e = \frac{c}{a}$ is called the *eccentricity of the hyperbola*. Since $c \geq a$, the eccentricity is never less than one. In terms of the eccentricity, the foci are at a distance of ae from the centre.

11.6.2 ***Standard equation of Hyperbola*** The equation of a hyperbola is simplest if the centre of the hyperbola is at the origin and the foci are on the x-axis or y-axis. The two such possible orientations are shown in Fig11.31.

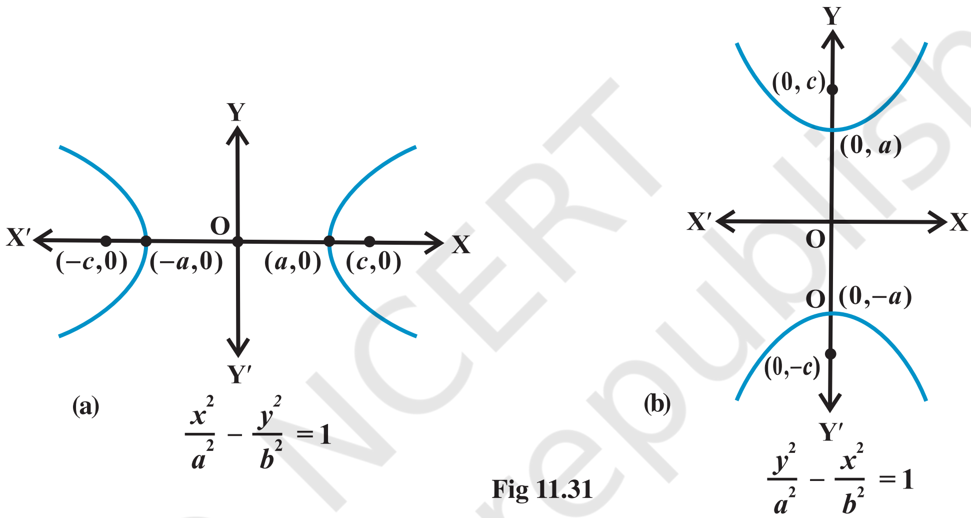


Fig 11.31

We will derive the equation for the hyperbola shown in Fig 11.31(a) with *foci* on the x-axis.

Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2. Let O be the origin and the line through O through F_2 be the positive x-axis and that through F_1 as the negative x-axis. The line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F_1 be $(-c,0)$ and F_2 be $(c,0)$ (Fig 11.32).

Let P(x, y) be any point on the hyperbola such that the difference of the distances from P to the farther point minus the closer point be $2a$. So given, $PF_1 - PF_2 = 2a$

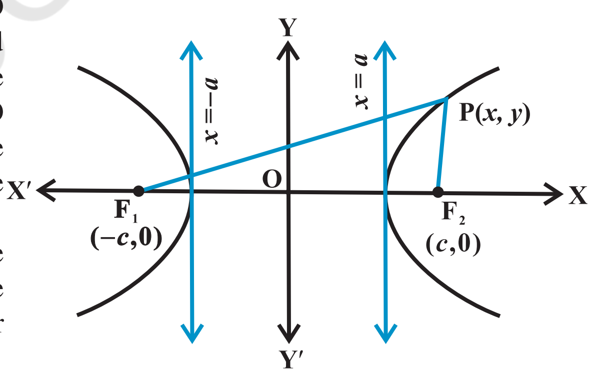


Fig 11.32

Using the distance formula, we have

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

i.e.,
$$\sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

Squaring both side, we get

$$(x + c)^2 + y^2 = 4a^2 + 4a \sqrt{(x-c)^2 + y^2} + (x - c)^2 + y^2$$

and on simplifying, we get

$$\frac{cx}{a} - a = \sqrt{(x-c)^2 + y^2}$$

On squaring again and further simplifying, we get

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

i.e.,
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{(Since } c^2 - a^2 = b^2\text{)}$$

Hence any point on the hyperbola satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Conversely, let P(x, y) satisfy the above equation with $0 < a < c$. Then

$$y^2 = b^2 \left(\frac{x^2 - a^2}{a^2}\right)$$

Therefore,
$$\text{PF}_1 = + \sqrt{(x+c)^2 + y^2}$$

$$= + \sqrt{(x+c)^2 + b^2\left(\frac{x^2 - a^2}{a^2}\right)} = a + \frac{c}{a}x$$

Similarly,
$$\text{PF}_2 = a - \frac{a}{c}x$$

In hyperbola $c > a$; and since P is to the right of the line $x = a$, $x > a$, $\frac{c}{a}x > a$. Therefore,

$a - \frac{c}{a}x$ becomes negative. Thus, $\text{PF}_2 = \frac{c}{a}x - a$.

Therefore $\quad PF_1 - PF_2 = a + \frac{c}{a}x - \frac{cx}{a} + a = 2a$

Also, note that if P is to the left of the line $x = -a$, then

$$PF_1 = -\left(a + \frac{c}{a}x\right), \ PF_2 = a - \frac{c}{a}x.$$

In that case $PF_2 - PF_1 = 2a$. So, any point that satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, lies on the hyperbola.

Thus, we proved that the equation of hyperbola with origin (0,0) and transverse axis along x-axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Note A hyperbola in which $a = b$ is called an *equilateral hyperbola*.

Discussion From the equation of the hyperbola we have obtained, it follows that, we have for every point (x, y) on the hyperbola, $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$.

i.e, $\left|\frac{x}{a}\right| \geq 1$, i.e., $x \leq -a$ or $x \geq a$. Therefore, no portion of the curve lies between the lines $x = +a$ and $x = -a$, (i.e. no real intercept on the conjugate axis).

Similarly, we can derive the equation of the hyperbola in Fig 11.31 (b) as $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

These two equations are known as the *standard equations* of *hyperbolas*.

Note The standard equations of hyperbolas have transverse and conjugate axes as the coordinate axes and the centre at the origin. However, there are hyperbolas with any two perpendicular lines as transverse and conjugate axes, but the study of such cases will be dealt in higher classes.

From the standard equations of hyperbolas (Fig11.29), we have the following observations:

1. Hyperbola is symmetric with respect to both the axes, since if (x, y) is a point on the hyperbola, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the hyperbola.

2. The foci are always on the transverse axis. It is the positive term whose denominator gives the transverse axis. For example, $\frac{x^2}{9} - \frac{y^2}{16} = 1$ has transverse axis along x-axis of length 6, while $\frac{y^2}{25} - \frac{x^2}{16} = 1$ has transverse axis along y-axis of length 10.

11.6.3 *Latus rectum*

Definition 9 Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.

As in ellipse, it is easy to show that the length of the latus rectum in hyperbola is $\frac{2b^2}{a}$.

Example 14 Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas:

(i) $\frac{x^2}{9} - \frac{y^2}{16} = 1$, (ii) $y^2 - 16x^2 = 16$

Solution **(i) Comparing the equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$ with the standard equation**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Here, $a = 3$, $b = 4$ and $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$

Therefore, the coordinates of the foci are $(\pm 5, 0)$ and that of vertices are $(\pm 3, 0)$.Also,

The eccentricity $e = \frac{c}{a} = \frac{5}{3}$. The latus rectum $= \frac{2b^2}{a} = \frac{32}{3}$

(ii) Dividing the equation by 16 on both sides, we have $\frac{y^2}{16} - \frac{x^2}{1} = 1$

Comparing the equation with the standard equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we find that

$a = 4$, $b = 1$ and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 1} = \sqrt{17}$.

Therefore, the coordinates of the foci are $(0, \pm\sqrt{17})$ and that of the vertices are $(0, \pm 4)$. Also,

The eccentricity $e = \frac{c}{a} = \frac{\sqrt{17}}{4}$. The latus rectum $= \frac{2b^2}{a} = \frac{1}{2}$.

Example 15 Find the equation of the hyperbola with foci $(0, \pm 3)$ and vertices $(0, \pm \frac{\sqrt{11}}{2})$.

Solution Since the foci is on y-axis, the equation of the hyperbola is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Since vertices are $(0, \pm \frac{\sqrt{11}}{2})$, $a = \frac{\sqrt{11}}{2}$

Also, since foci are $(0, \pm 3)$; $c = 3$ and $b^2 = c^2 - a^2 = \frac{25}{4}$.

Therefore, the equation of the hyperbola is

$$\frac{y^2}{\left(\frac{11}{4}\right)} - \frac{x^2}{\left(\frac{25}{4}\right)} = 1, \text{ i.e., } 100y^2 - 44x^2 = 275.$$

Example 16 Find the equation of the hyperbola where foci are $(0, \pm 12)$ and the length of the latus rectum is 36.

Solution Since foci are $(0, \pm 12)$, it follows that $c = 12$.

Length of the latus rectum $= \frac{2b^2}{a} = 36$ or $b^2 = 18a$

Therefore $\quad c^2 = a^2 + b^2$; gives

$\quad 144 = a^2 + 18a$

i.e., $\quad a^2 + 18a - 144 = 0$,

So $\quad a = -24, 6$.

Since a cannot be negative, we take $a = 6$ and so $b^2 = 108$.

Therefore, the equation of the required hyperbola is $\frac{y^2}{36} - \frac{x^2}{108} = 1$, i.e., $3y^2 - x^2 = 108$

EXERCISE 11.4

In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas.

1. $\frac{x^2}{16}-\frac{y^2}{9}=1$ **2.** $\frac{y^2}{9}-\frac{x^2}{27}=1$ **3.** $9y^2 - 4x^2 = 36$

4. $16x^2 - 9y^2 = 576$ **5.** $5y^2 - 9x^2 = 36$ **6.** $49y^2 - 16x^2 = 784$.

In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions.

7. Vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$

8. Vertices $(0, \pm 5)$, foci $(0, \pm 8)$

9. Vertices $(0, \pm 3)$, foci $(0, \pm 5)$

10. Foci $(\pm 5, 0)$, the transverse axis is of length 8.

11. Foci $(0, \pm 13)$, the conjugate axis is of length 24.

12. Foci $(\pm 3\sqrt{5}, 0)$, the latus rectum is of length 8.

13. Foci $(\pm 4, 0)$, the latus rectum is of length 12

14. vertices $(\pm 7, 0)$, $e = \frac{4}{3}$.

15. Foci $(0, \pm\sqrt{10})$, passing through (2,3)

Miscellaneous Examples

Example 17 The focus of a parabolic mirror as shown in Fig 11.33 is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB (Fig 11.33).

Solution Since the distance from the focus to the vertex is 5 cm. We have, $a = 5$. If the origin is taken at the vertex and the axis of the mirror lies along the positive x-axis, the equation of the parabolic section is

$$y^2 = 4\,(5)\,x = 20\,x$$

Note that $x = 45$. Thus

$$y^2 = 900$$

Therefore $y = \pm 30$

Hence $AB = 2y = 2 \times 30 = 60$ cm.

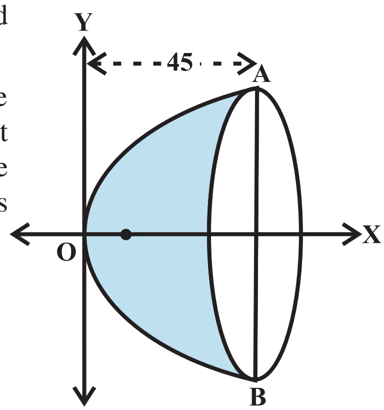


Fig 11.33

Example 18 A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there