**15. If the sum of p terms of an A.P. is q and the sum of q terms is p, show that the sum of p + q terms is – (p + q). Also, find the sum of first p – q terms (p > q).**

**Solution:**

The sum of n terms of an AP is given by

$S_n = \frac{n}{2}(2a + (n-1)d)$

Where a is the first term and d is the common difference

Given that $S_p = q$ and $S_q = p$

$\Rightarrow S_p = \frac{p}{2}(2a + (p-1)d)$ ....1

We know $S_p = q$

$\Rightarrow q = \frac{p}{2}(2a + (p-1)d)$

On rearranging we get

$\Rightarrow \frac{2q}{p} = 2a + (p-1)d \dots (i)$

$\Rightarrow S_q = \frac{q}{2}(2a + (q-1)d)$ .......2

Again we have $S_q = p$

$\Rightarrow p = \frac{q}{2}(2a + (q-1)d)$

On rearranging we get

$\Rightarrow \frac{2p}{q} = 2a + (q-1)d \dots (ii)$

Subtract (i) from (ii) that is (ii) – (i)

$\Rightarrow \frac{2p}{q} - \frac{2q}{p} = (q-1)d - (p-1)d$

Subtract (i) from (ii) that is (ii) – (i)

$\Rightarrow \frac{2p}{q} - \frac{2q}{p} = (q-1)d - (p-1)d$

On simplifying we get

$\Rightarrow 2\frac{p^2 - q^2}{pq} = (q - 1 - p + 1)d$

Using $a^2 - b^2 = (a + b)(a - b)$ formula we get

$\Rightarrow 2\frac{(p+q)(p-q)}{pq} = (q-p)d$

Computing and simplifying we get

$\Rightarrow 2\frac{-(p+q)(q-p)}{pq} = (q-p)d$

$\Rightarrow -2\frac{(p+q)}{pq} = d \ldots (iii)$

We have to show that $S_{p+q} = -(p+q)$

$S_{p+q} = \frac{p+q}{2}(2a + (p+q-1)d)$

Above equation can be written as

$= \frac{p}{2}(2a + (p+q-1)d) + \frac{q}{2}(2a + (p+q-1)d)$

$= \frac{p}{2}(2a + (p-1)d + qd) + \frac{q}{2}(2a + (q-1)d + pd)$

$= \frac{p}{2}(2a + (p-1)d) + \frac{pqd}{2} + \frac{q}{2}(2a + (q-1)d) + \frac{qpd}{2}$

Using (m) and (n)

$\Rightarrow S_{p+q} = S_p + S_q + p\,q\,d$

$= q + p + p\,q\,d$

Substitute d from (iii)

$\Rightarrow S_{p+q} = q + p + pq\left(-2\frac{(p+q)}{pq}\right)$

$= (p + q) - 2(p + q)$

$= -(p + q)$

Now we have to find sum of p – q terms that is $S_{p-q}$

$\Rightarrow S_{p-q} = \frac{p-q}{2}(2a + (p-q-1)d)$

$= \frac{p}{2}(2a + (p-q-1)d) - \frac{q}{2}(2a + (p-q-1)d)$

The above equation can be written as

$= \frac{p}{2}(2a + (p-1)d - qd) - \frac{q}{2}(2a + (p-1)d - qd)$

$$= \frac{p}{2}(2a + (p - q - 1)d) - \frac{q}{2}(2a + (p - q - 1)d)$$

The above equation can be written as

$$= \frac{p}{2}(2a + (p - 1)d - qd) - \frac{q}{2}(2a + (p - 1)d - qd)$$

$$= \frac{p}{2}(2a + (p - 1)d) - \frac{pqd}{2} - \frac{q}{2}(2a + (p - 1)d) + \frac{q^2 d}{2}$$

Using (m) and (n)

$$= S_p - \frac{pqd}{2} - \frac{q}{2}\frac{2S_p}{p} + \frac{q^2 d}{2}$$

Substituting the value of $S_p = q$ we get

$$= q - \frac{pqd}{2} - \frac{q^2}{p} + \frac{q^2 d}{2}$$

$$= q - \frac{q^2}{p} + \left(\frac{q^2 - qp}{2}\right)d$$

Substitute d from (iii)

$$= q - \frac{q^2}{p} + \left(\frac{q^2 - qp}{2}\right)\left(-2\frac{(p + q)}{pq}\right)$$

Simplifying and computing we get

$$= \frac{qp - q^2}{p} - (q^2 - qp)\left(\frac{p + q}{pq}\right)$$

$$= \frac{qp - q^2}{p} + (qp - q^2)\left(\frac{p + q}{pq}\right)$$

$$= \frac{qp - q^2}{p} + (qp - q^2)\left(\frac{1}{p} + \frac{1}{q}\right)$$

$$= \frac{qp - q^2}{p} + \frac{(qp - q^2)}{p} + \frac{(qp - q^2)}{q}$$

$$\Rightarrow S_{p-q} = 2\frac{q(p - q)}{p} + p - q$$

Hence sum of p – q terms is $2\frac{q(p-q)}{p} + p - q$