14. If $\theta_1, \theta_2, \theta_3, ..., \theta_n$ are in A.P., whose common difference is d, show that Sec θ_1 sec θ_2 + sec θ_2 sec θ_3 + ... + sec θ_{n-1} sec θ_n

$$= \frac{\tan\theta_n - \tan\theta_1}{\sin d}.$$

Solution:

Given $\theta_1, \theta_2, \theta_3, ..., \theta_n$ are in A.P., and common difference is d,
Now we have to prove that

Sec θ_1 sec θ_2 + sec θ_2 sec θ_3 + ... + sec θ_{n-1} sec θ_n = $\frac{\tan\theta_n - \tan\theta_1}{\sin d}$

On cross multiplication we get

⇒ Sin d (secθ_1 secθ_2 + secθ_2 secθ_3 + ... + sec θ_{n-1} sec θ_n) = tan θ_n – tan θ_1

We know sec x = 1/cos x using this formula we get

$$\Rightarrow \frac{\sin d}{\cos\theta_1\cos\theta_2} + \frac{\sin d}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin d}{\cos\theta_{n-1}\cos\theta_n} = \tan\theta_n - \tan\theta_1$$

Consider LHS

$$\Rightarrow \text{LHS} = \frac{\sin d}{\cos\theta_1\cos\theta_2} + \frac{\sin d}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin d}{\cos\theta_{n-1}\cos\theta_n}$$

Now we have to find value of d in terms of θ so that further simplification can be made

As $\theta_1, \theta_2, \theta_3, ..., \theta_n$ are in AP having common difference as d

Hence

$\theta_2 - \theta_1 = d$, $\theta_3 - \theta_2 = d$, ..., $\theta_n - \theta_{n-1} = d$

Take sin on both sides

Sin $(\theta_2 - \theta_1)$ = sin d, sin $(\theta_3 - \theta_2)$ = sin d, ..., sin $(\theta_n - \theta_{n-1})$ = sin d

Substitute appropriate value of sin d for each term in LHS

$$\Rightarrow \text{LHS} = \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1\cos\theta_2} + \frac{\sin(\theta_3 - \theta_2)}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin(\theta_n - \theta_{n-1})}{\cos\theta_{n-1}\cos\theta_n}$$

We know that sin (a – b) = sin a cos b – cos a sin b

Using this formula we get

$$\Rightarrow \text{LHS} = \frac{\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2} + \frac{\sin\theta_3\cos\theta_2 - \cos\theta_3\sin\theta_2}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin\theta_n\cos\theta_{n-1} - \cos\theta_n\sin\theta_{n-1}}{\cos\theta_{n-1}\cos\theta_n}$$

On simplifying we get

$$= \frac{\sin\theta_2\cos\theta_1}{\cos\theta_1\cos\theta_2} - \frac{\cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2} + \frac{\sin\theta_3\cos\theta_2}{\cos\theta_2\cos\theta_3} - \frac{\cos\theta_3\sin\theta_2}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin\theta_n\cos\theta_{n-1}}{\cos\theta_{n-1}\cos\theta_n} - \frac{\cos\theta_n\sin\theta_{n-1}}{\cos\theta_{n-1}\cos\theta_n}$$

We know that sin x/cos x = tan x

= tan θ_2 – tan θ_1 + tan θ_3 – tan θ_2 + ... + tan θ_n – tan θ_{n-1}

= - tan θ_1 + tan θ_n

= tan θ_n - tan θ_1

⇒ LHS = RHS

Hence proved