10. If $a_1, a_2, a_3, \ldots, a_n$ are in A.P., where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution:

Given $a_1, a_2, a_3, \ldots, a_n$ are in A.P., where $a_i > 0$ for all i

To prove that:

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

$$\Rightarrow \text{LHS}=\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

Multiplying the first term by $\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}-\sqrt{a_2}}$, the second term by $\frac{\sqrt{a_2}-\sqrt{a_3}}{\sqrt{a_2}-\sqrt{a_3}}$ and so on that is rationalizing each term

$$\Rightarrow \text{LHS}=\frac{1}{\sqrt{a_1}+\sqrt{a_2}}\times\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}-\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}\times\frac{\sqrt{a_2}-\sqrt{a_3}}{\sqrt{a_2}-\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}\times\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{\sqrt{a_{n-1}}-\sqrt{a_n}}$$

Now by using $(a + b)(a - b) = a^2 - b^2$

$$\Rightarrow \text{LHS}=\frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2}+\frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3}+\cdots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

As $a_1, a_2, a_3, \ldots, a_n$ are in AP let its common difference be d

$a_2 - a_1 = d$, $a_3 - a_2 = d$ ... $a_n - a_{n-1} = d$

Hence multiplying by -1

$a_1 - a_2 = -d$, $a_2 - a_3 = -d$ ... $a_n - a_{n-1} = -d$

Put these values in LHS

$$\Rightarrow \text{LHS}=\frac{\sqrt{a_1}-\sqrt{a_2}}{-d}+\frac{\sqrt{a_2}-\sqrt{a_3}}{-d}+\cdots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

$$=-\frac{1}{d}\left(\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+\cdots+\sqrt{a_{n-1}}-\sqrt{a_n}\right)$$

$$=-\frac{1}{d}\left(\sqrt{a_1}-\sqrt{a_n}\right)$$

To the above equation multiply and divide by $\left(\sqrt{a_1}+\sqrt{a_n}\right)$

$$\Rightarrow \text{LHS} = -\frac{1}{d}\frac{(\sqrt{a_1}-\sqrt{a_n})(\sqrt{a_1}+\sqrt{a_n})}{(\sqrt{a_1}+\sqrt{a_n})}$$

Using (a + b) (a – b) = $a^2 - b^2$

$$\Rightarrow \text{LHS} = -\frac{1}{d}\frac{a_1 - a_n}{(\sqrt{a_1}+\sqrt{a_n})}$$

The n^{th} term of AP is given by $t_n = a + (n - 1) d$

Where the $t_n = a_n$ is the last n^{th} term and $a = a_1$ is the first term

Hence $a_n = a_1 + (n - 1) d$

$\Rightarrow a_1 - a_n = - (n - 1) d$

Substitute $a_1 - a_n$ in LHS

$$\Rightarrow \text{LHS} = -\frac{1}{d}\frac{-(n-1)d}{(\sqrt{a_1}+\sqrt{a_n})}$$

$$= \frac{(n-1)}{(\sqrt{a_1}+\sqrt{a_n})}$$

$\Rightarrow$ LHS = RHS

Hence proved