

BINOMIAL THEOREM

$$\text{Consider } (x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2 \\ = x^2 + 2xy + y^2.$$

Here, xy can be written in two ways (xy and yx). Hence the coefficient of xy is equal to the number of ways x, y can be arranged, which is $2!$.

Thus in the expansion of $(x+y)^7$, the coefficient of x^3y^4 is equivalent to number of ways x, x, x, y, y, y, y can be arranged which is $7!/(3!4!) = {}^7C_3$.

Hence, in general

$$(x+y)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1} y + {}^n C_2 x^{n-2} y^2 + \dots + {}^n C_n y^n \text{ where } n \in \mathbb{N}$$

Note:

- This expansion has $(n+1)$ terms
- Its general term is given by $T_{r+1} = {}^n C_r x^{n-r} y^r$, where $r=0, 1, 2, 3, \dots, n$.
- In each term, the degree is n and the coefficient of $x^{n-r} y^r$ is equal to the number of ways $(n-r)$ x 's and r y 's can be arranged, which is given by

$$\frac{n!}{(n-r)! r!} = {}^n C_r$$

- $(p+1)^{\text{th}}$ term from the end is $(n-p+1)^{\text{th}}$ term from the beginning, i.e T_{n-p+1}

Properties of Binomial Coefficient

- Sum of two consecutive binomial coefficients,

$${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$$

$$\therefore {}^n C_r = {}^{n-1} C_{r-1}$$

- Ratio of two consecutive binomial coefficients,

$$\frac{{}^n C_r}{{}^n C_{r-1}} = \frac{n-r+1}{r}$$

- If ${}^n C_x = {}^n C_y$, then either $x=y$ or, $x+y=n$. So,

$${}^n C_x = {}^n C_{n-x} = \frac{n!}{x!(n-x)!}$$

Pascal's Triangle

Coefficient of binomial expansion can also be easily determined by Pascal's triangle.

$n=0$

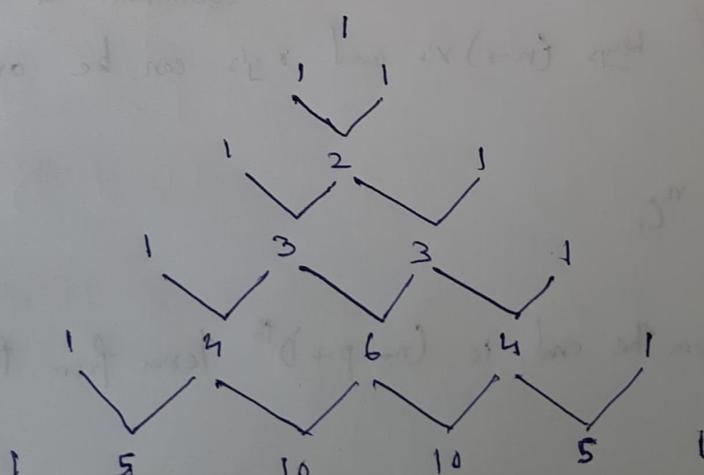
$n=1$

$n=2$

$n=3$

$n=4$

$n=5$



Construction of this triangle also justifies ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$, as

$$\begin{array}{ccccccc} n=3 & & 1 & & 3 = 3C_1 & & 3 = 3C_2 \\ & & & & \swarrow & & \\ n=4 & & 1 & 4 & & 6 = 4C_2 & \end{array}$$

Some Standard Expansions

We know that

$$(x+y)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1} y + {}^n C_2 x^{n-2} y^2 + \dots + {}^n C_n y^n, \text{ where } n \in \mathbb{N}$$

Putting $y=1$, we have

$$\begin{aligned}(1+x)^n &= {}^n C_0 x^n + {}^n C_1 x^{n-1} + {}^n C_2 x^{n-2} + \dots + {}^n C_n \\&= {}^n C_n x^n + {}^n C_{n-1} x^{n-1} + {}^n C_{n-2} x^{n-2} + \dots + {}^n C_0 \\&= {}^n C_0 + {}^n C_1 x^1 + {}^n C_2 x^2 + \dots + {}^n C_n x^n \\&= \sum_{r=0}^n {}^n C_r x^r\end{aligned}$$

In the above expansion, replacing x by $-x$, we have

$$\begin{aligned}(1-x)^n &= {}^n C_0 - {}^n C_1 x^1 + {}^n C_2 x^2 - \dots + (-1)^r \cdot {}^n C_r x^r + \dots \\&\quad + (-1)^n \cdot {}^n C_n x^n \\&= \sum_{r=0}^n {}^n C_r (-x)^r \\&= \sum_{r=0}^n (-1)^r {}^n C_r x^r\end{aligned}$$