BINOMIAL THEOREM

Consider $(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2$

$= x^2 + 2x + y^2$.

Here, xy can be written in two ways (xy and yx). Hence the coefficient of xy is equal to the number of ways x, y can be arranged, which is $2!$

Thus in the expansion of $(x+y)^7$, the coefficient of x^3y^4 is equivalent to number of ways x, x, x, y, y, y, y can be arranged which is $7!/(3!\,4!) = {}^7C_3$.

Hence, in general

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + \dots \dots {}^nC_n y^n \quad \text{where } n \in N$$

Note:

- This expansion has $(n+1)$ terms)
- Its general term is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$, where $r = 0, 1, 2, 3, \dots\dots, n$.
- In each term, the degree is n and the coefficient of $x^{n-r}y^r$ is equal to the number of ways $(n-r)$ x's and r y's can be arranged, which is given by

$$\frac{n!}{(n-r)!\,r!} = {}^nC_r$$

- $(p+1)^{th}$ term from the end is $(n-p+1)^{th}$ term from the beggining, i.e T_{n-p+1}

Properties of Binomial Coefficient

- Sum of two consequtive binomial coefficients,

$$^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$$

- $r\,{^nC_r} = n\,{^{n-1}C_{r-1}}$

- Ratio of two consequtive binomial coefficients,

$$\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$$

- If $^nC_x = {^nC_y}$, then either $x=y$ or $x+y=n$. So,

$$^nC_r = {^nC_{n-r}} = \frac{n!}{r!\,(n-r)!}$$

Pascal's Triangle

coefficient of binomial expansion can also be easily determined by Pascals triangle.

n=0   1

n=1   1   1

n=2   1   2   1

n=3   1   3   3   1

n=4   1   4   6   4   1

n=5   1   5   10   10   5   1

Construction of this triangle also justifies $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$, as

n=3   1 $3 = {^3C_1}$ $3 = {^3C_2}$

n=4   1   4 $6 = {^4C_2}$

Some Standard Expansions

We know that

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots$$
$$+ {}^nC_n y^n, \text{ where } n \in \mathbb{N}$$

Putting $y=1$ we have

$$(1+x)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \cdots + {}^nC_n$$
$$= {}^nC_n x^n + {}^nC_{n-1} x^{n-1} + {}^nC_{n-1} x^{n-1} + \cdots + {}^nC_0$$
$$= {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$$
$$= \sum_{r=0}^{n} {}^nC_r x^r$$

In the above expansion, replacing x by $-x$, we have

$$(1-x)^n = {}^nC_0 - {}^nC_1 x^1 + {}^nC_2 x^2 - \cdots + (-1)^r \cdot {}^nC_r x^r + \cdots$$
$$+ (-1)^n \, {}^nC_n x^n$$
$$= \sum_{r=0}^{n} {}^nC_r (-x)^r$$
$$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r x^r$$