22. $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

23. $\tan 4x = \dfrac{4\tan x\,(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$

24. $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

25. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

3.5 Trigonometric Equations

Equations involving trigonometric functions of a variable are called *trigonometric equations*. In this Section, we shall find the solutions of such equations. We have already learnt that the values of $\sin x$ and $\cos x$ repeat after an interval of 2π and the values of $\tan x$ repeat after an interval of π. The solutions of a trigonometric equation for which $0 \le x < 2\pi$ are called *principal solutions*. The expression involving integer 'n' which gives all solutions of a trigonometric equation is called the *general solution***.** We shall use '**Z**' to denote the set of integers.

The following examples will be helpful in solving trigonometric equations:

Example 18 Find the principal solutions of the equation $\sin x = \dfrac{\sqrt{3}}{2}$.

Solution We know that, $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ and $\sin\dfrac{2\pi}{3} = \sin\left(\pi - \dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.

Therefore, principal solutions are $x = \dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$.

Example 19 Find the principal solutions of the equation $\tan x = -\dfrac{1}{\sqrt{3}}$.

Solution We know that, $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$. Thus, $\tan\left(\pi - \dfrac{\pi}{6}\right) = -\tan\dfrac{\pi}{6} = -\dfrac{1}{\sqrt{3}}$

and $\tan\left(2\pi - \dfrac{\pi}{6}\right) = -\tan\dfrac{\pi}{6} = -\dfrac{1}{\sqrt{3}}$

Thus $\tan\dfrac{5\pi}{6} = \tan\dfrac{11\pi}{6} = -\dfrac{1}{\sqrt{3}}$.

Therefore, principal solutions are $\dfrac{5\pi}{6}$ and $\dfrac{11\pi}{6}$.

We will now find the general solutions of trigonometric equations. We have already seen that:

$\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$

$\cos x = 0$ gives $x = (2n+1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$.

We shall now prove the following results:

Theorem 1 For any real numbers x and y,

$$\sin x = \sin y \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}$$

Proof If $\sin x = \sin y$, then

$$\sin x - \sin y = 0 \text{ or } 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} = 0$$

which gives $\cos\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = (2n+1)\frac{\pi}{2}$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = (2n+1)\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = (2n+1)\pi + (-1)^{2n+1} y$ or $x = 2n\pi + (-1)^{2n} y$, where $n \in \mathbf{Z}$.

Combining these two results, we get

$$x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}.$$

Theorem 2 For any real numbers x and y, $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Proof If $\cos x = \cos y$, then

$$\cos x - \cos y = 0 \text{ i.e., } -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 0$$

Thus $\sin\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

Therefore $\frac{x+y}{2} = n\pi$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

i.e. $x = 2n\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

Hence $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Theorem 3 Prove that if x and y are not odd mulitple of $\frac{\pi}{2}$, then

$$\tan x = \tan y \text{ implies } x = n\pi + y, \text{ where } n \in \mathbf{Z}$$

Proof If $\tan x = \tan y$, then $\tan x - \tan y = 0$

or $$\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = 0$$

which gives $\sin (x - y) = 0$ (Why?)

Therefore $x - y = n\pi$, i.e., $x = n\pi + y$, where $n \in \mathbf{Z}$

Example 20 Find the solution of $\sin x = -\dfrac{\sqrt{3}}{2}$.

Solution We have $\sin x = -\dfrac{\sqrt{3}}{2} = -\sin\dfrac{\pi}{3} = \sin\left(\pi + \dfrac{\pi}{3}\right) = \sin\dfrac{4\pi}{3}$

Hence $\sin x = \sin\dfrac{4\pi}{3}$, which gives

$$x = n\pi + (-1)^n \frac{4\pi}{3}, \text{ where } n \in \mathbf{Z}.$$

> **Note** $\dfrac{4\pi}{3}$ is one such value of x for which $\sin x = -\dfrac{\sqrt{3}}{2}$. One may take any other value of x for which $\sin x = -\dfrac{\sqrt{3}}{2}$. The solutions obtained will be the same although these may apparently look different.

Example 21 Solve $\cos x = \dfrac{1}{2}$.

Solution We have, $\cos x = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

Therefore $x = 2n\pi \pm \dfrac{\pi}{3}$, where $n \in \mathbf{Z}$.

Example 22 Solve $\tan 2x = -\cot\left(x + \dfrac{\pi}{3}\right)$.

Solution We have, $\tan 2x = -\cot\left(x + \dfrac{\pi}{3}\right) = \tan\left(\dfrac{\pi}{2} + x + \dfrac{\pi}{3}\right)$

or $\qquad \tan 2x = \tan\left(x + \frac{5\pi}{6}\right)$

Therefore $\qquad 2x = n\pi + x + \frac{5\pi}{6}$, where $n \in \mathbf{Z}$

or $\qquad x = n\pi + \frac{5\pi}{6}$, where $n \in \mathbf{Z}$.

Example 23 Solve $\sin 2x - \sin 4x + \sin 6x = 0$.

Solution The equation can be written as

$$\sin 6x + \sin 2x - \sin 4x = 0$$

or $\qquad 2\sin 4x \cos 2x - \sin 4x = 0$

i.e. $\qquad \sin 4x(2\cos 2x - 1) = 0$

Therefore $\qquad \sin 4x = 0 \quad \text{or} \quad \cos 2x = \frac{1}{2}$

i.e. $\qquad \sin 4x = 0 \quad \text{or} \quad \cos 2x = \cos\frac{\pi}{3}$

Hence $\qquad 4x = n\pi \quad \text{or} \quad 2x = 2n\pi \pm \frac{\pi}{3}$, where $n \in \mathbf{Z}$

i.e. $\qquad x = \frac{n\pi}{4} \quad \text{or} \quad x = n\pi \pm \frac{\pi}{6}$, where $n \in \mathbf{Z}$.

Example 24 Solve $2\cos^2 x + 3\sin x = 0$

Solution The equation can be written as

$$2\left(1 - \sin^2 x\right) + 3\sin x = 0$$

or $\qquad 2\sin^2 x - 3\sin x - 2 = 0$

or $\qquad (2\sin x + 1)(\sin x - 2) = 0$

Hence $\qquad \sin x = -\frac{1}{2} \quad \text{or} \quad \sin x = 2$

But $\qquad \sin x = 2$ is not possible (Why?)

Therefore $\qquad \sin x = -\frac{1}{2} = \sin\frac{7\pi}{6}$.

Hence, the solution is given by

$$x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

EXERCISE 3.4

Find the principal and general solutions of the following equations:

1. $\tan x = \sqrt{3}$

2. $\sec x = 2$

3. $\cot x = -\sqrt{3}$

4. $\text{cosec } x = -2$

Find the general solution for each of the following equations:

5. $\cos 4x = \cos 2x$

6. $\cos 3x + \cos x - \cos 2x = 0$

7. $\sin 2x + \cos x = 0$

8. $\sec^2 2x = 1 - \tan 2x$

9. $\sin x + \sin 3x + \sin 5x = 0$

Miscellaneous Examples

Example 25 If $\sin x = \frac{3}{5}$, $\cos y = -\frac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin (x + y)$.

Solution We know that

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \quad \text{... (1)}$$

Now $\cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Therefore $\cos x = \pm \frac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\cos x = -\frac{4}{5}$

Now $\sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$

i.e. $\sin y = \pm \frac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \frac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get