24. If x = sec ϕ − tan ϕ and y = cosec ϕ + cot ϕ, then show that xy + x − y + 1 = 0.

[Hint: Find xy + 1 and then show tan x − y = −(xy + 1)]

Solution:

According to the question,

$x = \sec \phi - \tan \phi$ and $y = \operatorname{cosec} \phi + \cot \phi$

Given that, LHS = $xy + x - y + 1$

$$= (\sec\phi - \tan\phi)(\operatorname{cosec}\phi + \cot\phi) + (\sec\phi - \tan\phi) - (\operatorname{cosec}\phi + \cot\phi) + 1$$

$$= \sec\phi \operatorname{cosec}\phi + \cot\phi \sec\phi - \tan\phi \cot\phi - \tan\phi \operatorname{cosec}\phi + \sec\phi - \tan\phi - (\operatorname{cosec}\phi + \cot\phi) + 1$$

$$= \frac{1}{\sin\phi\cos\phi} + \frac{1}{\sin\phi} - 1 - \sec\phi + \sec\phi - \tan\phi - \left(\frac{1}{\sin\phi} + \frac{\cos\phi}{\sin\phi}\right) + 1$$

$$= \frac{1}{\sin\phi\cos\phi} + \frac{1}{\sin\phi} - \tan\phi - \left(\frac{1}{\sin\phi} + \frac{\cos\phi}{\sin\phi}\right)$$

$$= \frac{1}{\sin\phi\cos\phi} - \frac{\cos\phi}{\sin\phi} - \frac{\sin\phi}{\cos\phi}$$

$$= \frac{1}{\sin\phi\cos\phi} - \left(\frac{\cos\phi}{\sin\phi} + \frac{\sin\phi}{\cos\phi}\right)$$

$$= \frac{1}{\sin\phi\cos\phi} - \left(\frac{\cos^2\phi + \sin^2\phi}{\sin\phi\cos\phi}\right)$$

Since, $\sin^2\theta + \cos^2\theta = 1$

$$= \frac{1}{\sin\phi\cos\phi} - \left(\frac{1}{\sin\phi\cos\phi}\right) = 0$$

Thus, LHS = $xy + x - y + 1 = 0$