4. If $\cos(\alpha+\beta)=\frac{4}{5}$ and $\sin(\alpha-\beta)=\frac{5}{13}$, where α lie between 0 and π/4, find value of tan 2α

[Hint: Express tan 2α as tan (α + β + α − β)]

Solution:

According to the question,

$\cos(\alpha + \beta) = 4/5$ ...(i)

We know that,

$\sin x = \sqrt{1 - \cos^2 x}$

Therefore,

$\sin(\alpha + \beta) = \sqrt{1 - \cos^2(\alpha + \beta)}$

$\Rightarrow \sin(\alpha + \beta) = \sqrt{1 - (4/5)^2} = 3/5$ ...(ii)

Also,

$\sin(\alpha - \beta) = 5/13$ {given} ...(iii)

we know that,

$\cos x = \sqrt{1 - \sin^2 x}$

Therefore,

$\cos(\alpha - \beta) = \sqrt{1 - \sin^2(\alpha - \beta)}$

$\Rightarrow \cos(\alpha - \beta) = \sqrt{1 - (5/13)^2} = 12/13$ ...(iv)

Therefore,

$\tan 2\alpha = \tan(\alpha + \beta + \alpha - \beta)$

We know that,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\therefore \tan 2\alpha = \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$\Rightarrow \tan 2\alpha = \frac{\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}+\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}}{1-\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\times\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}}$$

From equation i, ii, iii and iv we have,

$$\Rightarrow \tan 2\alpha = \frac{\frac{\frac{3}{5}}{\frac{4}{5}} + \frac{\frac{5}{13}}{\frac{12}{13}}}{1 - \frac{\frac{3}{5}}{\frac{4}{5}} \times \frac{\frac{5}{13}}{\frac{12}{13}}}$$

$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}}$$

$$= \frac{\frac{9+5}{12}}{1 - \frac{15}{48}}$$

$$\Rightarrow \tan 2\alpha = \frac{14}{12\left(\frac{33}{48}\right)}$$

$$= \frac{56}{33}$$

Hence, tan 2α = 56/33