$\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$ are called *quadrantal angles*. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$$\cos 0^\circ = 1 \qquad \sin 0^\circ = 0,$$

$$\cos \frac{\pi}{2} = 0 \qquad \sin \frac{\pi}{2} = 1$$

$$\cos\pi = -1 \qquad \sin\pi = 0$$

$$\cos \frac{3\pi}{2} = 0 \qquad \sin \frac{3\pi}{2} = -1$$

$$\cos 2\pi = 1 \qquad \sin 2\pi = 0$$

Now, if we take one complete revolution from the point P, we again come back to same point P. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$$\sin (2n\pi + x) = \sin x,\ n \in \mathbf{Z},\ \ \cos (2n\pi + x) = \cos x,\ n \in \mathbf{Z}$$

Further, $\sin x = 0$, if $x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, ...$, i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, ...$ i.e., $\cos x$ vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus

$\sin x = 0$ implies $x = n\pi$, where n is any integer

$\cos x = 0$ implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions:

$$\text{cosec } x = \frac{1}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

$$\sec x = \frac{1}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\tan x = \frac{\sin x}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\cot x = \frac{\cos x}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

We have shown that for all real x, $\sin^2 x + \cos^2 x = 1$

It follows that

$$1 + \tan^2 x = \sec^2 x \qquad \text{(why?)}$$

$$1 + \cot^2 x = \text{cosec}^2 x \qquad \text{(why?)}$$

In earlier classes, we have discussed the values of trigonometric ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometric ratios studied in earlier classes. Thus, we have the following table:

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively.

3.3.1 *Sign of trigonometric functions*

Let P (a, b) be a point on the unit circle with centre at the origin such that $\angle AOP = x$. If $\angle AOQ = -x$, then the coordinates of the point Q will be $(a, -b)$ (Fig 3.7). Therefore

$$\cos(-x) = \cos x$$

and $$\sin(-x) = -\sin x$$

Since for every point P (a, b) on the unit circle, $-1 \le a \le 1$ and

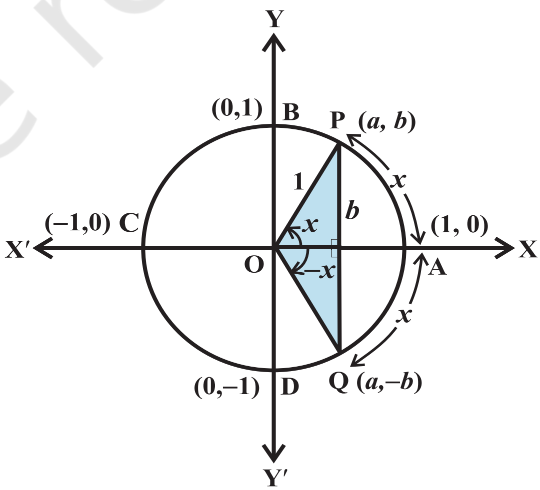


Fig 3.7

$-1 \le b \le 1$, we have $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$ for all x. We have learnt in previous classes that in the first quadrant ($0 < x < \frac{\pi}{2}$) a and b are both positive, in the second quadrant ($\frac{\pi}{2} < x < \pi$) a is negative and b is positive, in the third quadrant ($\pi < x < \frac{3\pi}{2}$) a and b are both negative and in the fourth quadrant ($\frac{3\pi}{2} < x < 2\pi$) a is positive and b is negative. Therefore, $\sin x$ is positive for $0 < x < \pi$, and negative for $\pi < x < 2\pi$. Similarly, $\cos x$ is positive for $0 < x < \frac{\pi}{2}$, negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$. Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\text{cosec } x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.3.2 ***Domain and range of trigonometric functions*** From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x,

$$-1 \le \sin x \le 1 \text{ and } -1 \le \cos x \le 1$$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval $[-1, 1]$, i.e., $-1 \le y \le 1$.

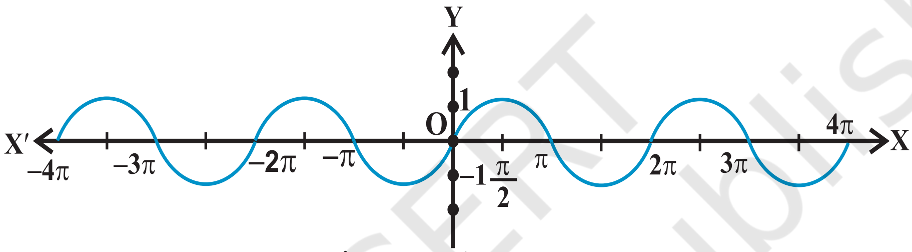
Since cosec $x = \frac{1}{\sin x}$, the domain of y = cosec x is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \geq 1$ or $y \leq -1\}$. Similarly, the domain of $y = \sec x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \leq -1$ or $y \geq 1\}$. The domain of $y = \tan x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set of all real numbers. The domain of $y = \cot x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and the range is the set of all real numbers.

We further observe that in the first quadrant, as x increases from 0 to $\frac{\pi}{2}$, $\sin x$ increases from 0 to 1, as x increases from $\frac{\pi}{2}$ to π, $\sin x$ decreases from 1 to 0. In the third quadrant, as x increases from π to $\frac{3\pi}{2}$, $\sin x$ decreases from 0 to –1and finally, in the fourth quadrant, $\sin x$ increases from –1 to 0 as x increases from $\frac{3\pi}{2}$ to 2π. Similarly, we can discuss the behaviour of other trigonometric functions. In fact, we have the following table:

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0
cos	decreases from 1 to 0	decreases from 0 to – 1	increases from –1 to 0	increases from 0 to 1
tan	increases from 0 to ∞	increases from –∞to 0	increases from 0 to ∞	increases from –∞to 0
cot	decreases from ∞ to 0	decreases from 0 to–∞	decreases from ∞ to 0	decreases from 0to –∞
sec	increases from 1 to ∞	increases from –∞to–1	decreases from –1to–∞	decreases from ∞ to 1
cosec	decreases from ∞ to 1	increases from 1 to ∞	increases from –∞to–1	decreases from–1to–∞

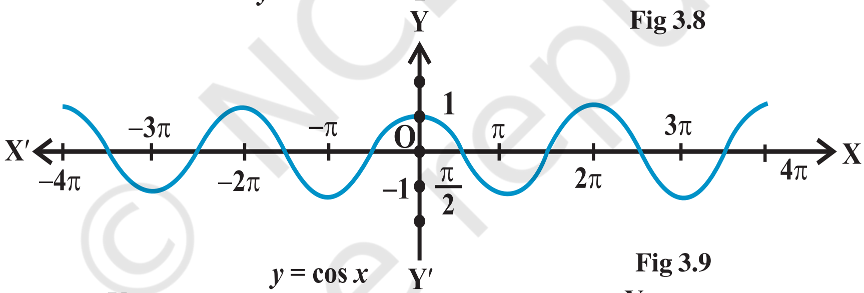
Remark In the above table, the statement $\tan x$ increases from 0 to ∞ (infinity) for $0 < x < \frac{\pi}{2}$ simply means that $\tan x$ increases as x increases for $0 < x < \frac{\pi}{2}$ and assumes arbitraily large positive values as x approaches to $\frac{\pi}{2}$. Similarly, to say that cosec x decreases from -1 to $-\infty$ (minus infinity) in the fourth quadrant means that cosec x decreases for $x \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as x approaches to 2π. The symbols ∞ and $-\infty$ simply specify certain types of behaviour of functions and variables.

We have already seen that values of sin x and cos x repeats after an interval of 2π. Hence, values of cosec x and sec x will also repeat after an interval of 2π. We



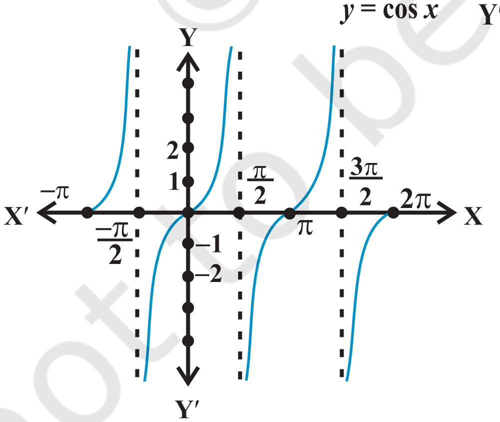
$y = \sin x$

Fig 3.8

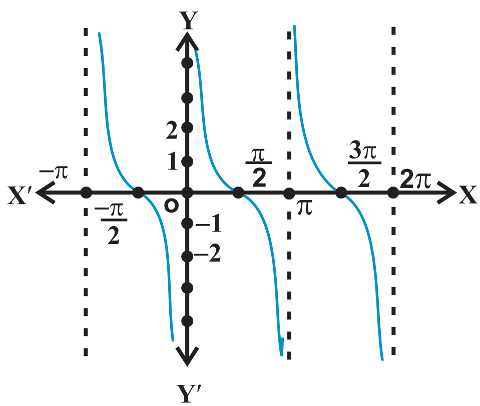


$y = \cos x$

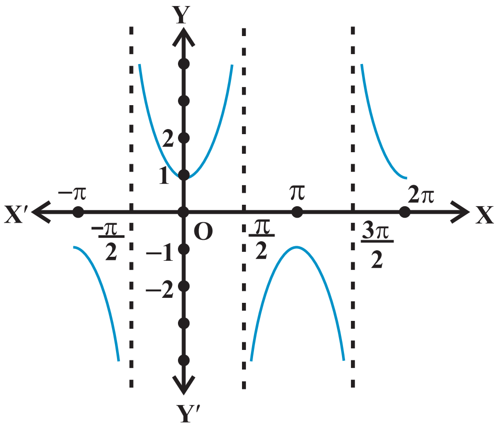
Fig 3.9



$y = \tan x$ Fig 3.10

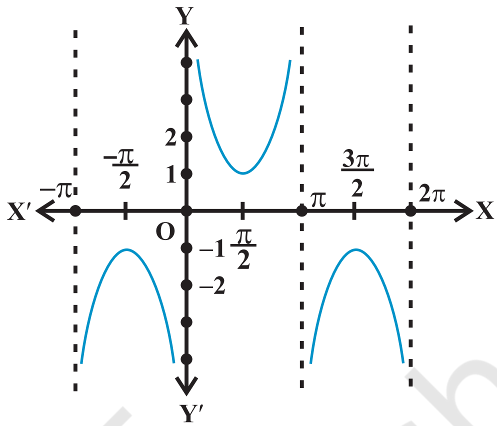


$y = \cot x$ Fig 3.11



$y = \sec x$

Fig 3.12



$y = \text{cosec}\, x$

Fig 3.13

shall see in the next section that tan $(\pi + x)$ = tan x. Hence, values of tan x will repeat after an interval of π. Since cot x is reciprocal of tan x, its values will also repeat after an interval of π. Using this knowledge and behaviour of trigonometic functions, we can sketch the graph of these functions. The graph of these functions are given above:

Example 6 If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now $\sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\sin x = \pm \frac{4}{5}$

Since x lies in third quadrant, $\sin x$ is negative. Therefore

$$\sin x = -\frac{4}{5}$$

which also gives

$$\text{cosec}\, x = -\frac{5}{4}$$

Further, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \text{ and } \cot x = \frac{\cos x}{\sin x} = \frac{3}{4}.$$

Example 7 If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution Since $\cot x = -\frac{5}{12}$, we have $\tan x = -\frac{12}{5}$

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$

Hence $\sec x = \pm \frac{13}{5}$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$$\sec x = -\frac{13}{5},$$

which also gives

$$\cos x = -\frac{5}{13}$$

Further, we have

$$\sin x = \tan x \cos x = (-\frac{12}{5}) \times (-\frac{5}{13}) = \frac{12}{13}$$

and $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}$.

Example 8 Find the value of $\sin \frac{31\pi}{3}$.

Solution We know that values of $\sin x$ repeats after an interval of 2π. Therefore

$$\sin \frac{31\pi}{3} = \sin (10\pi + \frac{\pi}{3}) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 9 Find the value of cos (–1710°).

Solution We know that values of cos x repeats after an interval of 2π or 360°. Therefore, cos (–1710°) = cos (–1710° + 5 × 360°)

= cos (–1710° + 1800°) = cos 90° = 0.

EXERCISE 3.2

Find the values of other five trigonometric functions in Exercises 1 to 5.

1. $\cos x = -\frac{1}{2}$, x lies in third quadrant.

2. $\sin x = \frac{3}{5}$, x lies in second quadrant.

3. $\cot x = \frac{3}{4}$, x lies in third quadrant.

4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.

5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Find the values of the trigonometric functions in Exercises 6 to 10.

6. sin 765°

7. cosec (– 1410°)

8. $\tan \frac{19\pi}{3}$

9. $\sin (-\frac{11\pi}{3})$

10. $\cot (-\frac{15\pi}{4})$

3.4 Trigonometric Functions of Sum and Difference of Two Angles

In this Section, we shall derive expressions for trigonometric functions of the sum and difference of two numbers (angles) and related expressions. The basic results in this connection are called *trigonometric identities*. We have seen that

1. $\sin (-x) = -\sin x$

2. $\cos (-x) = \cos x$

We shall now prove some more results:

3. cos $(x + y)$ = cos x cos y – sin x sin y

Consider the unit circle with centre at the origin. Let x be the angle P_4OP_1 and y be the angle P_1OP_2. Then $(x + y)$ is the angle P_4OP_2. Also let $(- y)$ be the angle P_4OP_3. Therefore, P_1, P_2, P_3 and P_4 will have the coordinates $P_1(\cos x, \sin x)$, P_2 $[\cos (x + y), \sin (x + y)]$, P_3 $[\cos (- y), \sin (- y)]$ and P_4 (1, 0) (Fig 3.14).

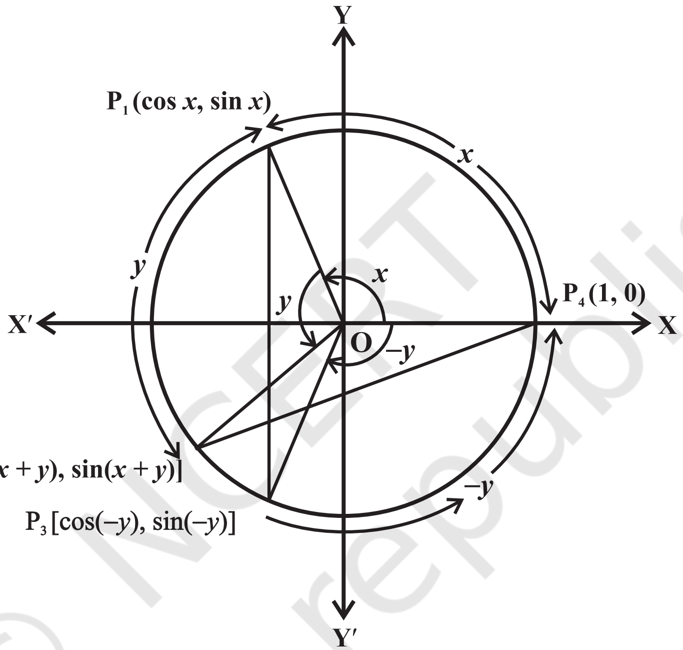


Fig 3.14

Consider the triangles P_1OP_3 and P_2OP_4. They are congruent (Why?). Therefore, P_1P_3 and P_2P_4 are equal. By using distance formula, we get

$$P_1P_3^{\,2} = [\cos x - \cos (- y)]^2 + [\sin x - \sin(-y]^2$$

$$= (\cos x - \cos y)^2 + (\sin x + \sin y)^2$$

$$= \cos^2 x + \cos^2 y - 2 \cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$$

$$= 2 - 2 (\cos x \cos y - \sin x \sin y) \qquad \text{(Why?)}$$

Also, $$P_2P_4^{\,2} = [1 - \cos (x + y)]^2 + [0 - \sin (x + y)]^2$$

$$= 1 - 2\cos (x + y) + \cos^2 (x + y) + \sin^2 (x + y)$$

$$= 2 - 2 \cos (x + y)$$

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$.

Therefore, $2 - 2(\cos x \cos y - \sin x \sin y) = 2 - 2\cos(x + y)$.

Hence $\cos(x + y) = \cos x \cos y - \sin x \sin y$

4. $\cos(x - y) = \cos x \cos y + \sin x \sin y$

Replacing y by $-y$ in identity 3, we get

$$\cos(x + (-y)) = \cos x \cos(-y) - \sin x \sin(-y)$$

or $$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

5. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

If we replace x by $\frac{\pi}{2}$ and y by x in Identity (4), we get

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = \sin x.$$

6. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

Using the Identity 5, we have

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \cos x.$$

7. $\sin(x + y) = \sin x \cos y + \cos x \sin y$

We know that

$$\sin(x + y) = \cos\left(\frac{\pi}{2} - (x + y)\right) = \cos\left(\left(\frac{\pi}{2} - x\right) - y\right)$$

$$= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y$$

$$= \sin x \cos y + \cos x \sin y$$

8. $\sin(x - y) = \sin x \cos y - \cos x \sin y$

If we replace y by $-y$, in the Identity 7, we get the result.

9. By taking suitable values of x and y in the identities 3, 4, 7 and 8, we get the following results:

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x \qquad \sin\left(\frac{\pi}{2} + x\right) = \cos x$$

$$\cos(\pi - x) = -\cos x \qquad \sin(\pi - x) = \sin x$$