Probability

* Basic Definitions:

1. Random experiment: An experiment whose outcome cannot be predicted with certainty.
(An experiment whose outcome is known in advance, is not a random experiment.). Random experiment has more than one outcome.

2. Sample Space: Set of all possible outcomes of a random experiment.
(A sample space having discrete number of sample points, is discrete sample space.).

3. Event: Subset of a sample space. (ϕ is also a subset of S which is called an impossible event. Sample space is the universal set for any random experiment.). (an event that cannot be further split is a simple event; an event consisting of more than one sample points is called compound event.).

4. Trial: Each performance of the random experiment is called a trial.

5. Equally likely events: Events are equally likely to occur. Chances of occurrence of equally likely events are equal.

6. Mutually exclusive events: If the occurrence of one of the events rules out the any of the remaining events.
A_1 & A_2 are mutually exclusive if $A_1 \cap A_2 = \phi$.
Events $A_1, A_2, \ldots, A_m$ are mutually exclusive if $A_i \cap A_j = \phi$ for all pairs (i,j) satisfying $1 \le i, j \le m$, where $i \ne j$

7. <u>Exhaustive Events</u> : A set of events is said to be exhaustive if atleast one of them must necessarily occur on each performance of the experiments. Events $A_1, A_2, \ldots, A_m$ are called exhaustive if

$$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_m = S, \text{ Sample Space ie. } \bigcup_{i=1}^{m} A_i = S$$

All the events collectively cover all possible outcomes of the experiment.

8. <u>Independent Events</u> : If occurence of any of the events does not affect the probability of occurrence of other events.

- Mutually exclusive events will never be independent & independent events will never be mutually exclusive.

* <u>Probability of an Event</u> : If there are n likely, mutually exclusive & exhaustive events & m of which are favourable to the event E, then probability of occurrence of event E, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{\text{no. of cases favourable to E}}{\text{Total no. of cases}} = \dfrac{m}{n}$.

$0 \leq m \leq n \Rightarrow 0 \leq P(E) \leq 1$.

No. of cases unfavourable to event $E = n-m$.

$P(E^c) = \dfrac{n-m}{n} = 1 - P(E) \Rightarrow P(E) + P(E^c) = 1$.

If E is sure event, $P(E) = 1$.

If E is impossible event, $P(E) = 0$.

Probability.

* **Odds in favour & against an event:**

If a cases are favourable to the event A & b cases are favourable to the $\bar{A}$, then

$$P(A) = \frac{a}{a+b} \quad \& \quad P(\bar{A}) = \frac{b}{a+b}.$$

Odds in favour of event A $= \dfrac{P(A)}{P(\bar{A})} = \dfrac{a}{b}$

Odds against event A $= \dfrac{P(\bar{A})}{P(A)} = \dfrac{b}{a}$.

* **Addition theorem of Probability:** If A & B are two events related with an experiment then the probability that either of the events will occur (or at least one of the events will occur) is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. where $P(A \cap B)$ is the event defined as the event that both A & B are occurring. For mutually exclusive events $P(A \cup B) = P(A) + P(B)$.

* Probability of each of n mutually exclusive, exhaustive & equally likely events is $1/n$.

* **Algebra of Events:**

i) $P(A) + P(A^c) = 1$

ii) $P(A-B) = P(A \cap \bar{B}) = P(A \cup B) - P(B) = P(A) - P(A \cap B)$

iii) $P(B-A) = P(B \cap \bar{A})$

iv) $P(A \Delta B) = P(A \cup B) - P(A \cap B)$.

v) $P(A \Delta B) = 2P(A \cup B) - (P(A) + P(B))$

vi) $P(A \Delta B) = P(A) + P(B) - 2P(A \cap B)$

vii) $P(\bar{A} \cap \bar{B}) = P(S) - P(A \cup B) = 1 - P(A \cup B)$.

viii) $A \subseteq B \Rightarrow P(A) \le P(B)$

ix) P(exactly two of A, B, C occur) =

$P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$.

x) P(at least two of A, B, C occur) =

$P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$.

* Conditional Probability: If A & B are two events, then the conditional probability of event A given that event B has already occurred $P(A/B)$ is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}.$$

Similarly, probability of B given that A has already occured will be $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$.

And $P(A \cap B) = P(B) \cdot P(A/B) = P(A) \cdot P(B/A)$.

$[P(B) \neq 0, P(A) \neq 0]$ – Multiplication Theorem

Generalised form –

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_m) = P(A_1)\, P(A_2/A_1)\, P(A_4/A_1 A_2 A_3) \cdots$$

$$\cdots P(A_m / A_1 A_2 \cdots A_{m-1}).$$

- If A & B are mutually independent, then $P(A/B) = P(A)$
- $P(A \cap B) = P(A) \cdot P(B)$ is the necessary & sufficient condition for the events A & B to be independent.

Probability

- 3 events A, B, C are said to be mutually independent if,

$P(A \cap B) = P(A) \cdot P(B)$; $P(B \cap C) = P(B) \cdot P(C)$; $P(C \cap A) = P(C) \cdot P(A)$.

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C).$$

They are pairwise independent if,

$P(A \cap B) = P(A) \cdot P(B)$; $P(B \cap C) = P(B) \cdot P(C)$; $P(C \cap A) = P(C) \cdot P(A)$.

* Total Probability Theorem: Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive & exhaustive events & event A is such that it can occur with any of the events $E_1, E_2, E_3, \ldots, E_n$ then the probability of occurrence of event A is $= P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$

$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \ldots + P(E_n) \cdot P(A/E_n)$.

$$P(A) = \sum_{i=1}^{n} P(E_i) P(A/E_i).$$

* Baye's Theorem: Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive & exhaustive events & A can occur with any of the events then given that A has already occurred with event E_i, then the conditional probability $P(E_i/A) =$

$$\frac{P(E_i \cap A)}{P(A)} = \frac{P(E_i) \cdot P(A/E_i)}{P(E_1) \cdot P(A/E_1) + \ldots + P(E_n) \cdot P(A/E_n)}$$

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}$$

* Binomial distribution for repeated experiments:

If the probability of success of an event in one trial is 'p', & that of its failure is 'q' so that $p+q=1$, then the probability of exactly r successes in n trials is.

- $P_r = {}^nC_r \, p^r q^{n-r}$

- Probability of success at least r times out of total n trials, $P(\geq r) =$

$${}^nC_r \, p^r q^{n-r} + {}^nC_{r+1} \, p^{r+1} q^{n-r-1} + \dots + {}^nC_n \, p^n$$

- Probability of success at most r times out of n trials, $P(\leq r) =$

$${}^nC_0 \, q^n + {}^nC_1 \, p^1 q^{n-1} + \dots + {}^nC_r \, p^r q^{n-r}$$

* Geometrical Probability:

$$\text{Probability} = \frac{\text{Measure of the favourable region}}{\text{Measure of the sample space.}}$$

* for 3 events A, B, C,

1. $P(\text{atleast one}) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$.
2. $P(\text{exactly one}) = P(A)+P(B)+P(C)-2P(A\cap B)-2P(B\cap C)-2P(C\cap A)+3P(A\cap B\cap C)$
3. $P(\text{atleast two}) = P(B\cap C)+P(C\cap A)+P(A\cap B)-2P(A\cap B\cap C)$
4. $P(\text{exactly two}) = P(B\cap C)+P(C\cap A)+P(A\cap B)-3P(A\cap B\cap C)$.