Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of $\vec{b}_1$ and $\vec{b}_2$, respectively. Then

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \text{ and } \vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

The given lines are coplanar if and only if $\overrightarrow{AB} \cdot \left(\vec{b}_1 \times \vec{b}_2\right) = 0$. In the cartesian form, it can be expressed as

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \qquad \text{... (4)}$$

Example 21 Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and } \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \text{ are coplanar.}$$

Solution Here, $x_1 = -3$, $y_1 = 1$, $z_1 = 5$, $a_1 = -3$, $b_1 = 1$, $c_1 = 5$
$x_2 = -1$, $y_2 = 2$, $z_2 = 5$, $a_2 = -1$, $b_2 = 2$, $c_2 = 5$

Now, consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

Therefore, lines are coplanar.

11.8 Angle between Two Planes

Definition 2 The angle between two planes is defined as the angle between their normals (Fig 11.18 (a)). Observe that if θ is an angle between the two planes, then so is 180 – θ (Fig 11.18 (b)). We shall take the acute angle as the angles between two planes.

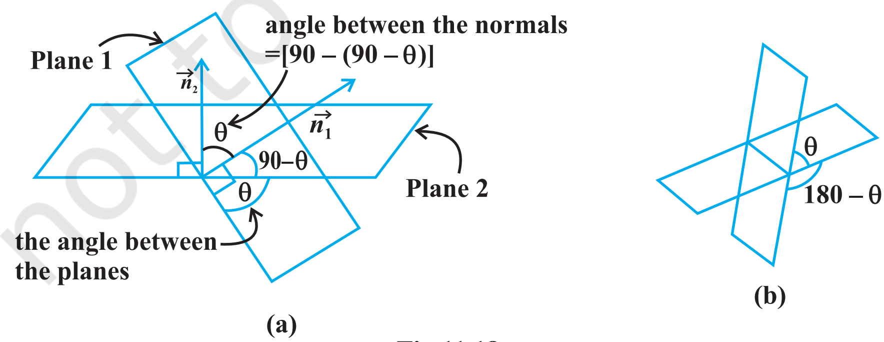

Fig 11.18

If $\vec{n}_1$ and $\vec{n}_2$ are normals to the planes and θ be the angle between the planes

$$\vec{r}\cdot\vec{n}_1 = d_1 \text{ and } \vec{r}\,.\,\vec{n}_2 = d_2.$$

Then θ is the angle between the normals to the planes drawn from some common point.

We have, $$\cos\theta = \left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right|$$

Note The planes are perpendicular to each other if $\vec{n}_1 . \vec{n}_2 = 0$ and parallel if $\vec{n}_1$ is parallel to $\vec{n}_2$.

Cartesian form Let θ be the angle between the planes,

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \text{ and } A_2 x + B_2 y + C_2 z + D_2 = 0$$

The direction ratios of the normal to the planes are A_1, B_1, C_1 and A_2, B_2, C_2 respectively.

Therefore, $$\cos\theta = \left|\frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

Note

1. If the planes are at right angles, then θ = 90° and so cos θ = 0. Hence, cos θ = $A_1A_2 + B_1B_2 + C_1C_2 = 0$.
2. If the planes are parallel, then $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.

Example 22 Find the angle between the two planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$ using vector method.

Solution The angle between two planes is the angle between their normals. From the equation of the planes, the normal vectors are

$$\vec{N}_1 = 2\hat{i} + \hat{j} - 2\hat{k} \text{ and } \vec{N}_2 = 3\hat{i} - 6\hat{j} - 2\hat{k}$$

Therefore $$\cos\theta = \left|\frac{\vec{N}_1\cdot\vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|}\right| = \left|\frac{(2\hat{i} + \hat{j} - 2\hat{k})\cdot(3\hat{i} - 6\hat{j} - 2\hat{k})}{\sqrt{4+1+4}\ \sqrt{9+36+4}}\right| = \left(\frac{4}{21}\right)$$

Hence $$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

Example 23 Find the angle between the two planes $3x - 6y + 2z = 7$ and $2x + 2y - 2z = 5$.

Solution Comparing the given equations of the planes with the equations

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \text{ and } A_2 x + B_2 y + C_2 z + D_2 = 0$$

We get
$$A_1 = 3, B_1 = -6, C_1 = 2$$
$$A_2 = 2, B_2 = 2, C_2 = -2$$

$$\cos\theta = \left|\frac{3\times 2 + (-6)(2) + (2)(-2)}{\sqrt{\left(3^2 + (-6)^2 + (-2)^2\right)}\sqrt{\left(2^2 + 2^2 + (-2)^2\right)}}\right|$$

$$= \left|\frac{-10}{7\times 2\sqrt{3}}\right| = \frac{5}{7\sqrt{3}} = \frac{5\sqrt{3}}{21}$$

Therefore, $\theta = \cos^{-1}\left(\frac{5\sqrt{3}}{21}\right)$

11.9 Distance of a Point from a Plane

Vector form

Consider a point P with position vector $\vec{a}$ and a plane π_1 whose equation is $\vec{r}\cdot\hat{n} = d$ (Fig 11.19).

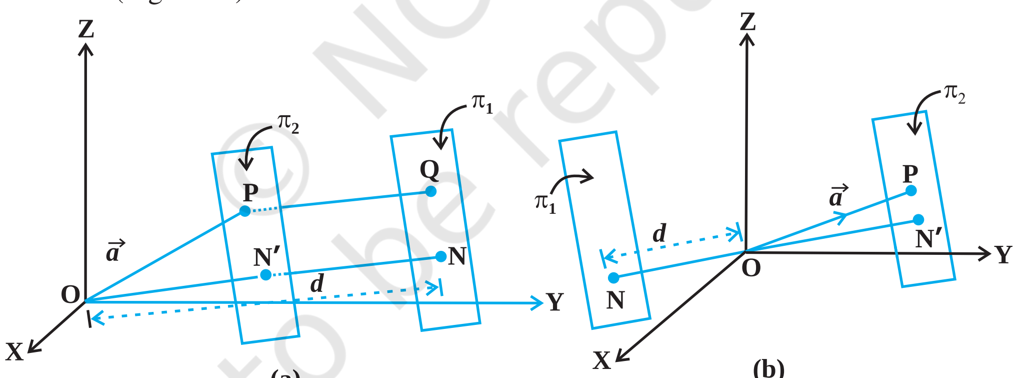

Fig 11.19

Consider a plane π_2 through P parallel to the plane π_1. The unit vector normal to π_2 is $\hat{n}$. Hence, its equation is $(\vec{r} - \vec{a})\cdot\hat{n} = 0$

i.e., $$\vec{r}\cdot\hat{n} = \vec{a}\cdot\hat{n}$$

Thus, the distance ON′ of this plane from the origin is $|\vec{a}\cdot\hat{n}|$. Therefore, the distance PQ from the plane π_1 is (Fig. 11.21 (a))

i.e., $$\text{ON} - \text{ON}' = |d - \vec{a}\cdot\hat{n}|$$

which is the length of the perpendicular from a point to the given plane.

We may establish the similar results for (Fig 11.19 (b)).

Note

1. If the equation of the plane π_2 is in the form $\vec{r} \cdot \vec{N} = d$, where $\vec{N}$ is normal to the plane, then the perpendicular distance is $\dfrac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$.

2. The length of the perpendicular from origin O to the plane $\vec{r} \cdot \vec{N} = d$ is $\dfrac{|d|}{|\vec{N}|}$ (since $\vec{a} = 0$).

Cartesian form

Let P(x_1, y_1, z_1) be the given point with position vector $\vec{a}$ and

$$Ax + By + Cz = D$$

be the Cartesian equation of the given plane. Then

$$\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$

$$\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

Hence, from Note 1, the perpendicular from P to the plane is

$$\left| \frac{(x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}) \cdot (A\hat{i} + B\hat{j} + C\hat{k}) - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

$$= \left| \frac{A x_1 + B y_1 + C z_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

Example 24 Find the distance of a point (2, 5, – 3) from the plane

$$\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$$

Solution Here, $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$, $\vec{N} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ and $d = 4$.

Therefore, the distance of the point (2, 5, – 3) from the given plane is

$$\frac{|(2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) - 4|}{|6\hat{i} - 3\hat{j} + 2\hat{k}|} = \frac{|12 - 15 - 6 - 4|}{\sqrt{36 + 9 + 4}} = \frac{13}{7}$$

11.10 Angle between a Line and a Plane

Definition 3 The angle between a line and a plane is the complement of the angle between the line and normal to the plane (Fig 11.20).

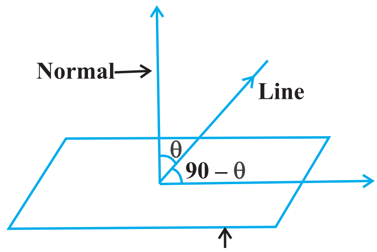

Fig 11.20

Vector form If the equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$ and the equation of the plane is $\vec{r} \cdot \vec{n} = d$. Then the angle θ between the line and the normal to the plane is

$$\cos\theta = \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}| \cdot |\vec{n}|} \right|$$

and so the angle φ between the line and the plane is given by 90 – θ, i.e.,

$$\sin(90 - \theta) = \cos\theta$$

i.e. $$\sin\phi = \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}|\ |\vec{n}|} \right| \text{ or } \phi = \sin^{-1}\left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \right|$$

Example 25 Find the angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$$

and the plane $10x + 2y - 11z = 3$.

Solution Let θ be the angle between the line and the normal to the plane. Converting the given equations into vector form, we have

$$\vec{r} = (-\hat{i} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $$\vec{r} \cdot (10\hat{i} + 2\hat{j} - 11\hat{k}) = 3$$

Here $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{n} = 10\hat{i} + 2\hat{j} - 11\hat{k}$

$$\sin\phi = \left| \frac{(2\hat{i} + 3\hat{j} + 6\hat{k}) \cdot (10\hat{i} + 2\hat{j} - 11\hat{k})}{\sqrt{2^2 + 3^2 + 6^2}\ \sqrt{10^2 + 2^2 + 11^2}} \right|$$

$$= \left| \frac{-40}{7 \times 15} \right| = \left| \frac{-8}{21} \right| = \frac{8}{21} \text{ or } \phi = \sin^{-1}\left(\frac{8}{21}\right)$$

EXERCISE 11.3

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z = 2$ (b) $x + y + z = 1$

(c) $2x + 3y - z = 5$ (d) $5y + 8 = 0$

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

3. Find the Cartesian equation of the following planes:

(a) $\vec{r}\cdot(\hat{i} + \hat{j} - \hat{k}) = 2$ (b) $\vec{r}\cdot(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

(c) $\vec{r}\cdot[(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(a) $2x + 3y + 4z - 12 = 0$ (b) $3y + 4z - 6 = 0$

(c) $x + y + z = 1$ (d) $5y + 8 = 0$

5. Find the vector and cartesian equations of the planes

(a) that passes through the point (1, 0, – 2) and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(b) that passes through the point (1,4, 6) and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

6. Find the equations of the planes that passes through three points.

(a) (1, 1, – 1), (6, 4, – 5), (– 4, – 2, 3)

(b) (1, 1, 0), (1, 2, 1), (– 2, 2, – 1)

7. Find the intercepts cut off by the plane $2x + y - z = 5$.

8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point (2, 2, 1).

10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$, $\vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point (2, 1, 3).

11. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

12. Find the angle between the planes whose vector equations are

$$\vec{r}\cdot(2\,\hat{i} + 2\,\hat{j} - 3\,\hat{k}) = 5 \text{ and } \vec{r}\cdot(3\,\hat{i} - 3\,\hat{j} + 5\,\hat{k}) = 3.$$

13. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

	Point	**Plane**
(a)	(0, 0, 0)	$3x - 4y + 12\,z = 3$
(b)	(3, – 2, 1)	$2x - y + 2z + 3 = 0$
(c)	(2, 3, – 5)	$x + 2y - 2z = 9$
(d)	(– 6, 0, 0)	$2x - 3y + 6z - 2 = 0$

Miscellaneous Examples

Example 26 A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Solution A cube is a rectangular parallelopiped having equal length, breadth and height. Let OADBFEGC be the cube with each side of length a units. (Fig 11.21)

The four diagonals are OE, AF, BG and CD.

The direction cosines of the diagonal OE which is the line joining two points O and E are

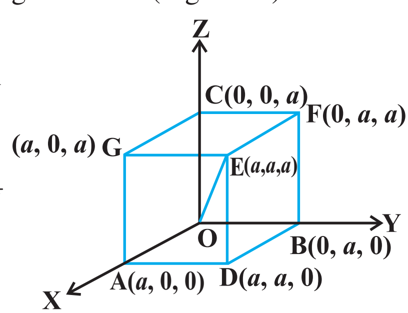

Fig 11.21

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

i.e., $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$