11.6 Plane

A plane is determined uniquely if any one of the following is known:

(i) the normal to the plane and its distance from the origin is given, i.e., equation of a plane in normal form.

(ii) it passes through a point and is perpendicular to a given direction.

(iii) it passes through three given non collinear points.

Now we shall find vector and Cartesian equations of the planes.

11.6.1 *Equation of a plane in normal form*

Consider a plane whose perpendicular distance from the origin is d ($d \neq 0$). Fig 11.10.

If $\overrightarrow{\text{ON}}$ is the normal from the origin to the plane, and $\hat{n}$ is the unit normal vector along $\overrightarrow{\text{ON}}$. Then $\overrightarrow{\text{ON}} = d\,\hat{n}$. Let P be any point on the plane. Therefore, $\overrightarrow{\text{NP}}$ is perpendicular to $\overrightarrow{\text{ON}}$.

Therefore, $\overrightarrow{\text{NP}} \cdot \overrightarrow{\text{ON}} = 0$ ... (1)

Let $\vec{r}$ be the position vector of the point P, then $\overrightarrow{\text{NP}} = \vec{r} - d\,\hat{n}$ (as $\overrightarrow{\text{ON}} + \overrightarrow{\text{NP}} = \overrightarrow{\text{OP}}$)

Therefore, (1) becomes

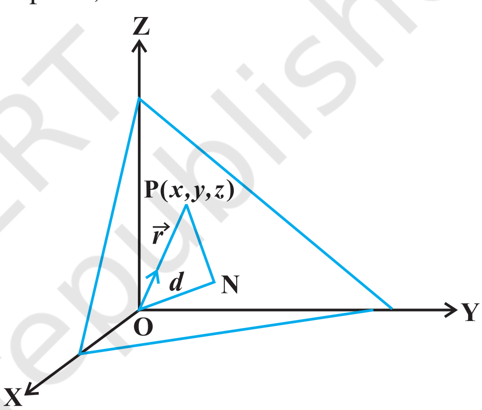

Fig 11.10

$$(\vec{r} - d\,\hat{n}) \cdot d\,\hat{n} = 0$$

or $$(\vec{r} - d\,\hat{n}) \cdot \hat{n} = 0 \qquad (d \neq 0)$$

or $$\vec{r} \cdot \hat{n} - d\,\hat{n} \cdot \hat{n} = 0$$

i.e., $$\boldsymbol{\vec{r} \cdot \hat{n} = d} \qquad (\text{as } \hat{n} \cdot \hat{n} = 1) \qquad \ldots (2)$$

This is the vector form of the equation of the plane.

Cartesian form

Equation (2) gives the vector equation of a plane, where $\hat{n}$ is the unit vector normal to the plane. Let P(x, y, z) be any point on the plane. Then

$$\overrightarrow{\text{OP}} = \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Let l, m, n be the direction cosines of $\hat{n}$. Then

$$\hat{n} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$$

Therefore, (2) gives

$$(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) \cdot (l\,\hat{i} + m\,\hat{j} + n\,\hat{k}) = d$$

i.e., $$\boldsymbol{lx + my + nz = d} \qquad \text{... (3)}$$

This is the cartesian equation of the plane in the normal form.

> **Note** Equation (3) shows that if $\vec{r} \cdot (a\,\hat{i} + b\,\hat{j} + c\,\hat{k}) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the Cartesian equation of the plane, where a, b and c are the direction ratios of the normal to the plane.

Example 13 Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$. Also find its cartesian form.

Solution Let $\vec{n} = 2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}$. Then

$$\hat{n} = \frac{\vec{n}}{|\,\vec{n}\,|} = \frac{2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}}{\sqrt{4+9+16}} = \frac{2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}}{\sqrt{29}}$$

Hence, the required equation of the plane is

$$\vec{r} \cdot \left(\frac{2}{\sqrt{29}}\,\hat{i} + \frac{-3}{\sqrt{29}}\,\hat{j} + \frac{4}{\sqrt{29}}\,\hat{k} \right) = \frac{6}{\sqrt{29}}$$

Example 14 Find the direction cosines of the unit vector perpendicular to the plane $\vec{r} \cdot (6\,\hat{i} - 3\,\hat{j} - 2\,\hat{k}) + 1 = 0$ passing through the origin.

Solution The given equation can be written as

$$\vec{r} \cdot (-6\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}) = 1 \qquad \text{... (1)}$$

Now $$|-6\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}| = \sqrt{36+9+4} = 7$$

Therefore, dividing both sides of (1) by 7, we get

$$\vec{r} \cdot \left(-\frac{6}{7}\,\hat{i} + \frac{3}{7}\,\hat{j} + \frac{2}{7}\,\hat{k} \right) = \frac{1}{7}$$

which is the equation of the plane in the form $\vec{r} \cdot \hat{n} = d$.

This shows that $\hat{n} = -\frac{6}{7}\,\hat{i} + \frac{3}{7}\,\hat{j} + \frac{2}{7}\,\hat{k}$ is a unit vector perpendicular to the plane through the origin. Hence, the direction cosines of $\hat{n}$ are $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$.

Example 15 Find the distance of the plane $2x - 3y + 4z - 6 = 0$ from the origin.

Solution Since the direction ratios of the normal to the plane are 2, –3, 4; the direction cosines of it are

$$\frac{2}{\sqrt{2^2+(-3)^2+4^2}}, \frac{-3}{\sqrt{2^2+(-3)^2+4^2}}, \frac{4}{\sqrt{2^2+(-3)^2+4^2}}, \text{ i.e., } \frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

Hence, dividing the equation $2x-3y+4z-6=0$ i.e., $2x-3y+4z=6$ throughout by $\sqrt{29}$, we get

$$\frac{2}{\sqrt{29}}x + \frac{-3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

This is of the form $lx + my + nz = d$, where d is the distance of the plane from the origin. So, the distance of the plane from the origin is $\frac{6}{\sqrt{29}}$.

Example 16 Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x - 3y + 4z - 6 = 0$.

Solution Let the coordinates of the foot of the perpendicular P from the origin to the plane is (x_1, y_1, z_1) (Fig 11.11).

Then, the direction ratios of the line OP are x_1, y_1, z_1.

Writing the equation of the plane in the normal form, we have

$$\frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

where, $\frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$ are the direction cosines of the OP.

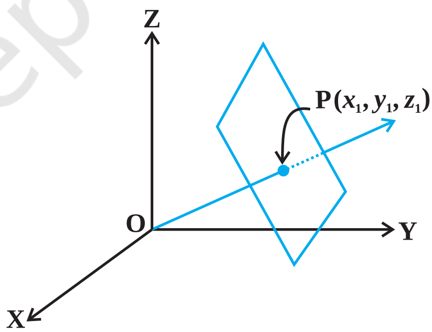

Fig 11.11

Since *d.c.'s* and direction ratios of a line are proportional, we have

$$\frac{x_1}{\frac{2}{\sqrt{29}}} = \frac{y_1}{\frac{-3}{\sqrt{29}}} = \frac{z_1}{\frac{4}{\sqrt{29}}} = k$$

i.e., $$x_1 = \frac{2k}{\sqrt{29}}, y_1 = \frac{-3k}{\sqrt{29}}, z_1 = \frac{4k}{\sqrt{29}}$$

Substituting these in the equation of the plane, we get $k = \dfrac{6}{\sqrt{29}}$.

Hence, the foot of the perpendicular is $\left(\dfrac{12}{29}, \dfrac{-18}{29}, \dfrac{24}{29}\right)$.

Note If d is the distance from the origin and l, m, n are the direction cosines of the normal to the plane through the origin, then the foot of the perpendicular is (ld, md, nd).

11.6.2 *Equation of a plane perpendicular to a given vector and passing through a given point*

In the space, there can be many planes that are perpendicular to the given vector, but through a given point $P(x_1, y_1, z_1)$, only one such plane exists (see Fig 11.12).

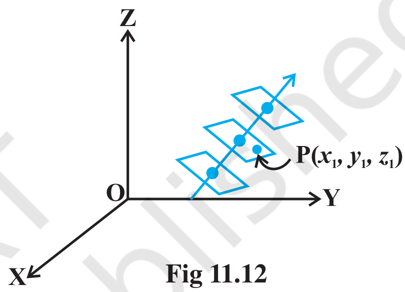

Fig 11.12

Let a plane pass through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$.

Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ in the plane. (Fig 11.13).

Then the point P lies in the plane if and only if $\overrightarrow{AP}$ is perpendicular to $\vec{N}$. i.e., $\overrightarrow{AP} . \vec{N} = 0$. But $\overrightarrow{AP} = \vec{r} - \vec{a}$. Therefore, $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$ ... (1)

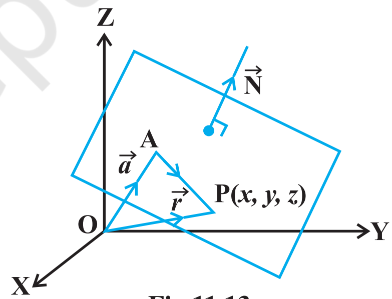

Fig 11.13

This is the vector equation of the plane.

Cartesian form

Let the given point A be (x_1, y_1, z_1), P be (x, y, z) and direction ratios of $\vec{N}$ are A, B and C. Then,

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ and } \vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

Now $\quad (\vec{r} - \vec{a}) \cdot \vec{N} = 0$

So $\quad \left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] \cdot (A\hat{i} + B\hat{j} + C\hat{k}) = 0$

i.e. $\quad \mathbf{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}$

Example 17 Find the vector and cartesian equations of the plane which passes through the point (5, 2, – 4) and perpendicular to the line with direction ratios 2, 3, – 1.

Solution We have the position vector of point (5, 2, – 4) as $\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and the normal vector $\vec{N}$ perpendicular to the plane as $\vec{N} = 2\hat{i} + 3\hat{j} - \hat{k}$

Therefore, the vector equation of the plane is given by $(\vec{r} - \vec{a}).\vec{N} = 0$

or $$[\vec{r} - (5\hat{i} + 2\hat{j} - 4\hat{k})] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0 \quad \text{... (1)}$$

Transforming (1) into Cartesian form, we have

$$[(x-5)\hat{i} + (y-2)\hat{j} + (z+4)\hat{k}] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0$$

or $$2(x-5) + 3(y-2) - 1(z+4) = 0$$

i.e. $$2x + 3y - z = 20$$

which is the cartesian equation of the plane.

11.6.3 *Equation of a plane passing through three non collinear points*

Let R, S and T be three non collinear points on the plane with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively (Fig 11.14).

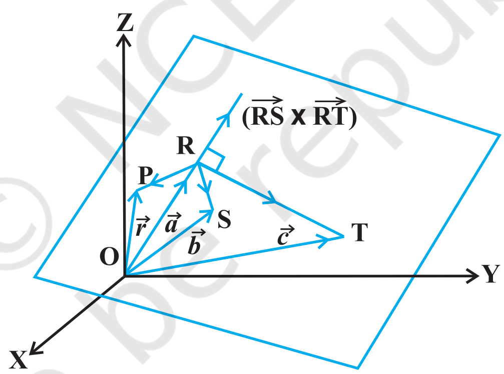

Fig 11.14

The vectors $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are in the given plane. Therefore, the vector $\overrightarrow{RS} \times \overrightarrow{RT}$ is perpendicular to the plane containing points R, S and T. Let $\vec{r}$ be the position vector of any point P in the plane. Therefore, the equation of the plane passing through R and perpendicular to the vector $\overrightarrow{RS} \times \overrightarrow{RT}$ is

$$(\vec{r} - \vec{a}) \cdot (\overrightarrow{RS} \times \overrightarrow{RT}) = 0$$

or $$(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0 \quad \text{... (1)}$$

This is the equation of the plane in vector form passing through three noncollinear points.

Note Why was it necessary to say that the three points had to be non collinear? If the three points were on the same line, then there will be many planes that will contain them (Fig 11.15).

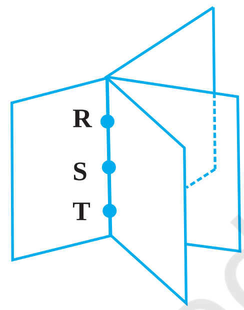

Fig 11.15

These planes will resemble the pages of a book where the line containing the points R, S and T are members in the binding of the book.

Cartesian form

Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of the points R, S and T respectively. Let (x, y, z) be the coordinates of any point P on the plane with position vector $\vec{r}$. Then

$$\overrightarrow{RP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$

$$\overrightarrow{RS} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\overrightarrow{RT} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$

Substituting these values in equation (1) of the vector form and expressing it in the form of a determinant, we have

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

which is the equation of the plane in Cartesian form passing through three non collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Example 18 Find the vector equations of the plane passing through the points R(2, 5, – 3), S(– 2, – 3, 5) and T(5, 3,– 3).

Solution Let $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$, $\vec{b} = -2\hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{c} = 5\hat{i} + 3\hat{j} - 3\hat{k}$

Then the vector equation of the plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is given by

$$(\vec{r} - \vec{a}) \cdot (\overrightarrow{RS} \times \overrightarrow{RT}) = 0 \quad \text{(Why?)}$$

or $$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$

i.e. $$[\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})] \cdot [(-4\hat{i} - 8\hat{j} + 8\hat{k}) \times (3\hat{i} - 2\hat{j})] = 0$$

11.6.4 *Intercept form of the equation of a plane*

In this section, we shall deduce the equation of a plane in terms of the intercepts made by the plane on the coordinate axes. Let the equation of the plane be

$$\text{A}x + \text{B}y + \text{C}z + \text{D} = 0 \quad (\text{D} \neq 0) \qquad \text{... (1)}$$

Let the plane make intercepts a, b, c on x, y and z axes, respectively (Fig 11.16).

Hence, the plane meets x, y and z-axes at $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, respectively.

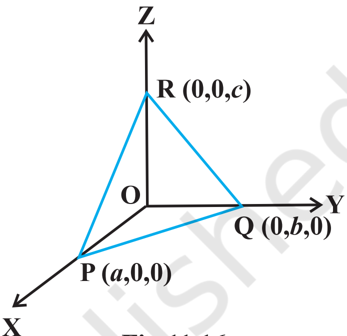

Fig 11.16

Therefore $\quad \text{A}a + \text{D} = 0$ or $\text{A} = \dfrac{-\text{D}}{a}$

$\text{B}b + \text{D} = 0$ or $\text{B} = \dfrac{-\text{D}}{b}$

$\text{C}c + \text{D} = 0$ or $\text{C} = \dfrac{-\text{D}}{c}$

Substituting these values in the equation (1) of the plane and simplifying, we get

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{... (1)}$$

which is the required equation of the plane in the intercept form.

Example 19 Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z-axis respectively.

Solution Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{... (1)}$$

Here $\quad a = 2, b = 3, c = 4.$

Substituting the values of a, b and c in (1), we get the required equation of the plane as $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{4} = 1$ or $6x + 4y + 3z = 12$.

11.6.5 *Plane passing through the intersection of two given planes*

Let π_1 and π_2 be two planes with equations $\vec{r} \cdot \hat{n}_1 = d_1$ and $\vec{r} \cdot \hat{n}_2 = d_2$ respectively. The position vector of any point on the line of intersection must satisfy both the equations (Fig 11.17).

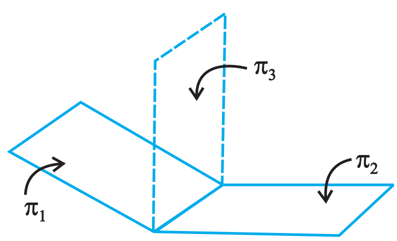

Fig 11.17

If $\vec{t}$ is the position vector of a point on the line, then

$$\vec{t} \cdot \hat{n}_1 = d_1 \text{ and } \vec{t} \cdot \hat{n}_2 = d_2$$

Therefore, for all real values of λ, we have

$$\vec{t} \cdot (\hat{n}_1 + \lambda \hat{n}_2) = d_1 + \lambda d_2$$

Since $\vec{t}$ is arbitrary, it satisfies for any point on the line.

Hence, the equation $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$ represents a plane π_3 which is such that if any vector $\vec{r}$ satisfies both the equations π_1 and π_2, it also satisfies the equation π_3 i.e., any plane passing through the intersection of the planes

$$\vec{r} \cdot \vec{n}_1 = d_1 \text{ and } \vec{r} \cdot \vec{n}_2 = d_2$$

has the equation $\quad \boldsymbol{\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2}$ ... (1)

Cartesian form

In Cartesian system, let

$$\vec{n}_1 = A_1 \hat{i} + B_2 \hat{j} + C_1 \hat{k}$$

$$\vec{n}_2 = A_2 \hat{i} + B_2 \hat{j} + C_2 \hat{k}$$

and $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then (1) becomes

$$x(A_1 + \lambda A_2) + y(B_1 + \lambda B_2) + z(C_1 + \lambda C_2) = d_1 + \lambda d_2$$

or $\quad \boldsymbol{(A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0}$ ... (2)

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

Example 20 Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 6$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = -5$, and the point (1, 1, 1).

Solution Here, $\vec{n}_1 = \hat{i} + \hat{j} + \hat{k}$ and $\vec{n}_2 = 2\hat{i} + 3\hat{j} + 4\hat{k}$;

and $\quad d_1 = 6$ and $d_2 = -5$

Hence, using the relation $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$, we get

$$\vec{r} \cdot [\hat{i} + \hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})] = 6 - 5\lambda$$

or $\quad \vec{r} \cdot [(1+2\lambda)\hat{i} + (1+3\lambda)\hat{j} + (1+4\lambda)\hat{k}] = 6 - 5\lambda$ ... (1)

where, λ is some real number.

Taking $\vec{r} = x\hat{i} + y\,\hat{j} + z\,\hat{k}$, we get

$$(x\hat{i} + y\,\hat{j} + z\,\hat{k})\cdot[(1+2\lambda)\,\hat{i} + (1+3\lambda)\,\hat{j} + (1+4\lambda)\hat{k}] = 6 - 5\lambda$$

or $\quad (1 + 2\lambda)\, x + (1 + 3\lambda)\, y + (1 + 4\lambda)\, z = 6 - 5\lambda$

or $\quad (x + y + z - 6) + \lambda\,(2x + 3y + 4z + 5) = 0 \quad \text{... (2)}$

Given that the plane passes through the point (1,1,1), it must satisfy (2), i.e.

$$(1 + 1 + 1 - 6) + \lambda\,(2 + 3 + 4 + 5) = 0$$

or $\quad \lambda = \dfrac{3}{14}$

Putting the values of λ in (1), we get

$$\vec{r}\left[\left(1+\frac{3}{7}\right)\hat{i} + \left(1+\frac{9}{14}\right)\hat{j} + \left(1+\frac{6}{7}\right)\hat{k}\right] = 6 - \frac{15}{14}$$

or $\quad \vec{r}\left(\dfrac{10}{7}\hat{i} + \dfrac{23}{14}\hat{j} + \dfrac{13}{7}\hat{k}\right) = \dfrac{69}{14}$

or $\quad \vec{r}\cdot(20\hat{i} + 23\,\hat{j} + 26\hat{k}) = 69$

which is the required vector equation of the plane.

11.7 Coplanarity of Two Lines

Let the given lines be

$$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1 \quad \text{... (1)}$$

and $\quad \vec{r} = \vec{a}_2 + \mu\vec{b}_2 \quad \text{... (2)}$

The line (1) passes through the point, say A, with position vector $\vec{a}_1$ and is parallel to $\vec{b}_1$. The line (2) passes through the point, say B with position vector $\vec{a}_2$ and is parallel to $\vec{b}_2$.

Thus, $\quad \overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$

The given lines are coplanar if and only if $\overrightarrow{AB}$ is perpendicular to $\vec{b}_1 \times \vec{b}_2$.

i.e. $\quad \overrightarrow{AB}.(\vec{b}_1 \times \vec{b}_2) = 0$ or $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$

Cartesian form

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the points A and B respectively.

Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of $\vec{b}_1$ and $\vec{b}_2$, respectively. Then

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \text{ and } \vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

The given lines are coplanar if and only if $\overrightarrow{AB} \cdot (\vec{b}_1 \times \vec{b}_2) = 0$. In the cartesian form, it can be expressed as

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \qquad \text{... (4)}$$

Example 21 Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and } \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \text{ are coplanar.}$$

Solution Here, $x_1 = -3$, $y_1 = 1$, $z_1 = 5$, $a_1 = -3$, $b_1 = 1$, $c_1 = 5$

$x_2 = -1$, $y_2 = 2$, $z_2 = 5$, $a_2 = -1$, $b_2 = 2$, $c_2 = 5$

Now, consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

Therefore, lines are coplanar.

11.8 Angle between Two Planes

Definition 2 The angle between two planes is defined as the angle between their normals (Fig 11.18 (a)). Observe that if θ is an angle between the two planes, then so is 180 – θ (Fig 11.18 (b)). We shall take the acute angle as the angles between two planes.

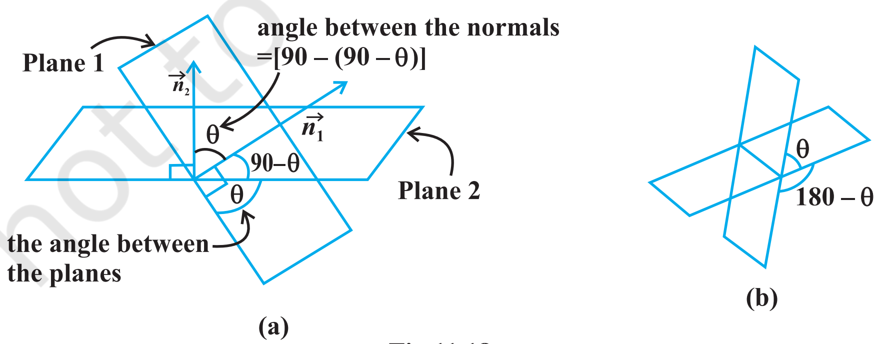

Fig 11.18