23. The plane *ax* + *by* = 0 is rotated about its line of intersection with the plane *z* = 0 through an angle α. Prove that the equation of the plane in its new position is

$$ax + by \pm (\sqrt{a^2 + b^2}\tan\alpha)z = 0.$$

Solution:

Given planes are: ax + by = 0 …. (i) and z = 0 …. (ii)

Now, the equation of any plan passing through the line of intersection of plane (i) and (ii) is

(ax + by) + kz = 0 ⇒ ax + by + kz = 0 …. (iii)

Dividing both sides by $\sqrt{a^2+b^2+k^2}$, we get

$$\frac{a}{\sqrt{a^2+b^2+k^2}}x+\frac{b}{\sqrt{a^2+b^2+k^2}}y+\frac{k}{\sqrt{a^2+b^2+k^2}}z=0$$

So, direction cosines of the normal to the plane are

$$\frac{a}{\sqrt{a^2+b^2+k^2}},\ \frac{b}{\sqrt{a^2+b^2+k^2}},\ \frac{k}{\sqrt{a^2+b^2+k^2}}$$

and the direction cosines of the plane (*i*) are

$$\frac{a}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}},\ 0$$

As, α is the angle between the planes (*i*) and (*iii*), we get

$$\Rightarrow \cos\alpha=\frac{a.a+b.b+k.0}{\sqrt{a^2+b^2+k^2}.\sqrt{a^2+b^2}}$$

$$\cos\alpha=\frac{a^2+b^2}{\sqrt{a^2+b^2+k^2}.\sqrt{a^2+b^2}}$$

$$\cos\alpha=\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+k^2}} \quad \Rightarrow \cos^2\alpha=\frac{a^2+b^2}{a^2+b^2+k^2}$$

$$(a^2+b^2+k^2)\cos^2\alpha=a^2+b^2$$

$$a^2\cos^2\alpha+b^2\cos^2\alpha+k^2\cos^2\alpha=a^2+b^2$$

$$k^2\cos^2\alpha=a^2-a^2\cos^2\alpha+b^2-b^2\cos^2\alpha$$

$$k^2\cos^2\alpha=\alpha^2(1-\cos^2\alpha)+b^2(1-\cos^2\alpha)$$

$$k^2\cos^2\alpha=a^2\sin^2\alpha+b^2\sin^2\alpha$$

$$k^2\cos^2\alpha=(a^2+b^2)\sin^2\alpha$$

$$\Rightarrow k^2=(a^2+b^2)\frac{\sin^2\alpha}{\cos^2\alpha} \quad \Rightarrow k=\pm\sqrt{a^2+b^2}.\tan\alpha$$

Putting the value of k in eq. (*iii*) we get

$ax+by\pm\left(\sqrt{a^2+b^2}.\tan\alpha\right)z=0$ which is the required equation of plane.

- Hence proved.

Dividing both sides by $\sqrt{a^2+b^2+k^2}$, we get

$$\frac{a}{\sqrt{a^2+b^2+k^2}}x+\frac{b}{\sqrt{a^2+b^2+k^2}}y+\frac{k}{\sqrt{a^2+b^2+k^2}}z=0$$

So, direction cosines of the normal to the plane are

$$\frac{a}{\sqrt{a^2+b^2+k^2}},\ \frac{b}{\sqrt{a^2+b^2+k^2}},\ \frac{k}{\sqrt{a^2+b^2+k^2}}$$

and the direction cosines of the plane (*i*) are

$$\frac{a}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}},0$$

As, α is the angle between the planes (*i*) and (*iii*), we get

$$\Rightarrow \cos\alpha=\frac{a.a+b.b+k.0}{\sqrt{a^2+b^2+k^2}.\sqrt{a^2+b^2}}$$

$$\cos\alpha=\frac{a^2+b^2}{\sqrt{a^2+b^2+k^2}.\sqrt{a^2+b^2}}$$

$$\cos\alpha=\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+k^2}} \quad \Rightarrow \cos^2\alpha=\frac{a^2+b^2}{a^2+b^2+k^2}$$

$$(a^2+b^2+k^2)\cos^2\alpha=a^2+b^2$$

$$a^2\cos^2\alpha+b^2\cos^2\alpha+k^2\cos^2\alpha=a^2+b^2$$

$$k^2\cos^2\alpha=a^2-a^2\cos^2\alpha+b^2-b^2\cos^2\alpha$$

$$k^2\cos^2\alpha=\alpha^2(1-\cos^2\alpha)+b^2(1-\cos^2\alpha)$$

$$k^2\cos^2\alpha=a^2\sin^2\alpha+b^2\sin^2\alpha$$

$$k^2\cos^2\alpha=(a^2+b^2)\sin^2\alpha$$

$$\Rightarrow k^2=(a^2+b^2)\frac{\sin^2\alpha}{\cos^2\alpha} \quad \Rightarrow k=\pm\sqrt{a^2+b^2}.\tan\alpha$$

Putting the value of k in eq. (*iii*) we get

$ax+by\pm\left(\sqrt{a^2+b^2}.\tan\alpha\right)z=0$ which is the required equation of plane.

- Hence proved.