The nth Roots of Unity

The nth roots of unity, it means any complex number z, which satisfies the equation z = 1 or $z = (1)^{1/n}$

or $z = \cos(2k\pi/n) + i\sin(2k\pi/n)$, where k = 0, 1, 2, ... , (n — 1)

Properties of nth Roots of Unity

1. nth roots of unity form a GP with common ratio $e^{(i2\pi/n)}$.
2. Sum of nth roots of unity is always 0.
3. Sum of nth powers of nth roots of unity is zero, if p is a multiple of n
4. Sum of pth powers of nth roots of unity is zero, if p is not a multiple of n.
5. Sum of pth powers of nth roots of unity is n, ifp is a multiple of n.
6. Product of nth roots of unity is $(-1)^{(n-1)}$.
7. The nth roots of unity lie on the unit circle |z| = 1 and divide its circumference into n equal parts.

The Cube Roots of Unity

Cube roots of unity are 1, ω, ω ,

where $\omega = -1/2 + i\sqrt{3}/2 = e^{(i2\pi/3)}$ and $\omega = (-1 - i\sqrt{3})/2$

$\omega^{3r+1} = \omega$, $\omega^{3r+2} = \omega^{2}$

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^{2r} =$
0, if r is not a multiple of 3.
3, if r is,a multiple of 3.

(ii) $\omega^3 = \omega^{3r} = 1$

(iii) $\omega^{3r+1} = \omega$, $\omega^{3r+2} = \omega^2$

(iv) Cube roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into 3 equal parts.

(v) It always forms an equilateral triangle.

(vi) Cube roots of – 1 are -1, $-\omega$, $-\omega^2$.

Important Identities

(i) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

(ii) $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

(iii) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(iv) $x^2 - xy + y^2 = (x + \omega y)(x + y\omega^2)$

(v) $x^2 + y^2 = (x + iy)(x - iy)$

(vi) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(vii) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

(viii) $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

or $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$

or $(x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$

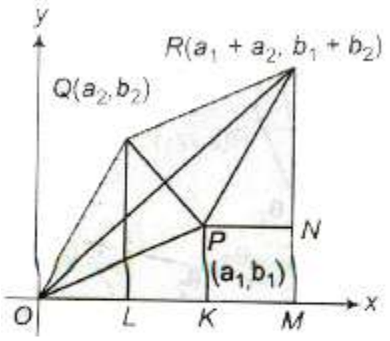
(ix) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

Geometrical Representations of Complex Numbers

1. Geometrical Representation of Addition

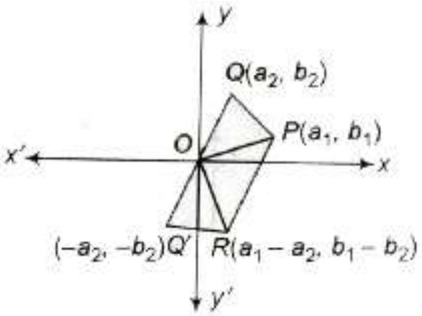
If two points P and Q represent complex numbers z_1 and z_2 respectively, in the Argand plane, then the sum $z_1 + z_2$ is represented

by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.



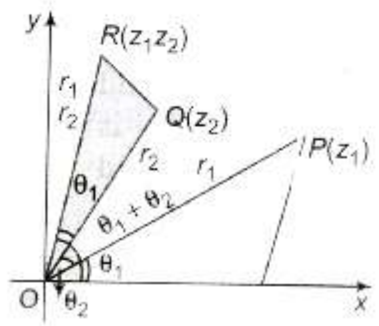
2. Geometrical Representation of Subtraction

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ia_2$ be two complex numbers represented by points P (a_1, b_1) and Q(a_2, b_2) in the Argand plane. Q' represents the complex number (—z_2). Complete the parallelogram OPRQ' by taking OP and OQ' as two adjacent sides.



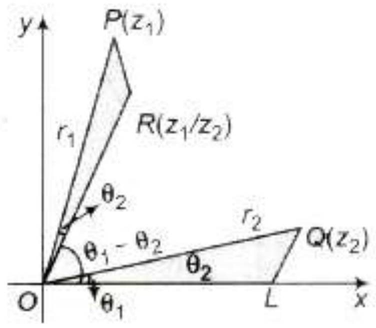
The sum of z_1 and —z_2 is represented by the extremity R of the diagonal OR of parallelogram OPRQ'. R represents the complex number z_1 — z_2.

3. Geometrical Representation of Multiplication of Complex Numbers



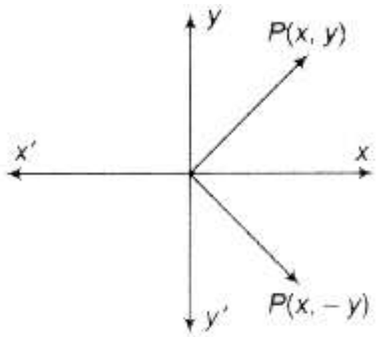
R has the polar coordinates (r_1r_2, $\theta_1 + \theta_2$) and it represents the complex numbers z_1z_2.

4. Geometrical Representation of the Division of Complex Numbers



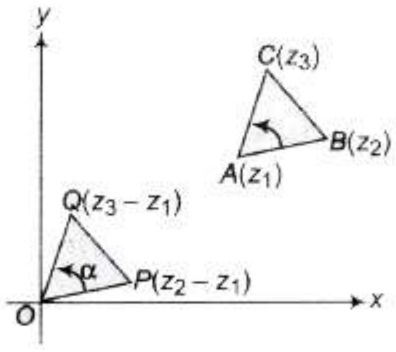
R has the polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$ and it represents the complex number z_1/z_2.
|z|=|z| and arg (z) = – arg (z). The general value of arg (z) is $2n\pi$ – arg (z).

If a point P represents a complex number z, then its conjugate i is represented by the image of P in the real axis.

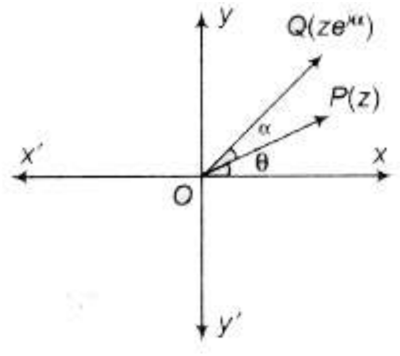


Concept of Rotation

Let z_1, z_2 and z_3 be the vertices of a ΔABC described in anti-clockwise sense. Draw OP and OQ parallel and equal to AB and AC, respectively. Then, point P is $z_2 - z_1$ and Q is $z_3 - z_1$. If OP is rotated through angle a in anti-clockwise, sense it coincides with OQ.



Important Points to be Remembered



(a) $ze^{i\alpha}$ a is the complex number whose modulus is r and argument $\theta + \alpha$.
(b) Multiplication by $e^{-i\alpha}$ to z rotates the vector OP in clockwise sense through an angle α.

(ii) If z_1, z_2, z_3 and z_4 are the affixes of the points A, B,C and D, respectively in the Argand plane.

(a) AB is inclined to CD at the angle arg $[(z_2 – z_1)/(z_4 – z_3)]$.

(b) If CD is inclines at 90° to AB, then arg $[(z_2 – z_1)/(z_4 – z_3)] = \pm(\pi/2)$.

(c) If z_1 and z_2 are fixed complex numbers, then the locus of a point z satisfying arg $[([(z – z_1)/(z – z_2)] = \pm(\pi/2)$.

Logarithm of a Complex Number

Let $z = x + iy$ be a complex number and in polar form of z is $re^{i\theta}$, then

$\log(x + iy) = \log (re^{i\theta}) = \log (r) + i\theta$

$\log(\sqrt{x}^2 + y^2) + i\tan^{-1} (y/x)$

or $\log(z) = \log (|z|)+ i\text{amp} (z)$,

In general,

$z = re^{i(\theta + 2n\pi)}$

$\log z = \log|z| + i\arg z + 2n\pi i$

Applications of Complex Numbers in Coordinate Geometry

Distance between complex Points

(i) Distance between $A(z_1)$ and $B(_1)$ is given by

$AB = |z_2 — z_1| = \sqrt{(x_2 + x_1)^2 + (y_2 + y_1)^2}$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

(ii) The point P (z) which divides the join of segment AB in the ratio m : n is given by

$z = (mz_2 + nz_1)/(m + n)$

If P divides the line externally in the ratio m : n, then

$z = (mz_2 – nz_1)/(m – n)$

Triangle in Complex Plane

(i) Let ABC be a triangle with vertices A (z_1), B(z_2) and C(z_3) then

(a) Centroid of the ΔABC is given by

$z = 1/3(z_1 + z_2 + z_3)$

(b) Incentre of the AABC is given by

$z = (az_1 + bz_2 + cz_3)/(a + b + c)$

(ii) Area of the triangle with vertices A(z_1), B(z_2) and C(z_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

For an equilateral triangle,

$z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$

(iii) The triangle whose vertices are the points represented by complex numbers z_1, z_2 and z_3 is equilateral, if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

i.e., $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Straight Line in Complex Plane

(i) The general equation of a straight line is az + az + b = 0, where a is a complex number and b is a real number.

(ii) The complex and real slopes of the line az + az are -a/a and – i[(a + a)/(a – a)].

(iii) The equation of straight line through z_1 and z_2 is $z = tz_1 + (1 — t)z_2$, where t is real.

(iv) If z_1 and z_2 are two fixed points, then $|z — z_1| = z — z_2|$ represents perpendicular bisector of the line segment joining z1 and z2.

(v) Three points z_1, z_2 and z_3 are collinear, if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

This is also, the equation of the line passing through $_1$, z_2 and z_3 and slope is defined to be $(z_1 – z_2)/z_1 – z_2$

(vi) Length of Perpendicular The length of perpendicular from a point z_1 to az + az + b = 0 is given by $|az_1 + az_1 + b|/2|a|$

(vii) arg $(z – z_1)/(z – z_2) = \beta$

Locus is the arc of a circle which the segment joining z_1 and z_2 as a chord.

(viii) The equation of a line parallel to the line az + az + b = 0 is az + az + λ = 0, where λ ∈ R.

(ix) The equation of a line parallel to the line az + az + b = 0 is az + az + iλ = 0, where λ ∈ R.

(x) If z_1 and z_2 are two fixed points, then I z — z11 =I z z21 represents perpendicular bisector of the segment joining A(z1) and B(z2).

(xi) The equation of a line perpendicular to the plane $z(z_1 – z_2) + z(z_1 – z_2) = |z_1|^2 – |z_2|^2$.

(xii) If z_1, z_2 and z_3 are the affixes of the points A, B and C in the Argand plane, then

(a) ∠BAC = $\arg[(z_3 – z_1/z_2 – z_1)]$

(b) $[(z_3 – z_1)/(z_2 – z_1)] = |z_3 – z_1|/|z_2 – z_1|$ (cos α + isin α), where α = ∠BAC.

(xiii) If z is a variable point in the argand plane such that arg (z) = θ, then locus of z is a straight line through the origin inclined at an angle θ with X-axis.

(xiv) If z is a variable point and z_1 is fixed point in the argand plane such that $(z — z_1)= \theta$, then locus of z is a straight line passing through the point z_1 and inclined at an angle θ with the X-axis.

(xv) If z is a variable point and z_1, z_2 are two fixed points in the Argand plane, then

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

Locus of z is the line segment joining z_1 and z_2.

(b) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

Locus of z is a straight line joining z_1 and z_2 but z does not lie between z1 and z_2.

(c) $\arg[(z - z_1)/(z - z_2)] = 0$ or π

Locus z is a straight line passing through z_1 and z_2.

(d) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

Locus of z is a circle with z_1 and z_2 as the extremities of diameter.

Circle in Complete Plane

(i) An equation of the circle with centre at z_0 and radius r is

$|z - z_0| = r$

or $zz - z_0z - z_0z + z_0$

- $|z - z_0| < r$, represents interior of the circle.
- $|z - z_0| > r$, represents exterior of the circle.
- $|z - z_0| \leq r$ is the set of points lying inside and on the circle $|z - z_0| = r$. Similarly, $|z - z_0| \geq r$ is the set of points lying outside and on the circle $|z - z_0| = r$.
- **General equation of a circle is**

$zz - az - az + b = 0$

where a is a complex number and b is a real number. Centre of the circle = – a

Radius of the circle = $\sqrt{aa - b}$ or $\sqrt{|a|^2 - b}$

(a) Four points z_1, z_2, z_3 and z_4 are concyclic, if

$[(z_4 - z_1)(z_2 - z_3)]/[(z_4 - z_3)(z_2 - z_1)]$ is purely real.

(ii) $|z - z_1|/|z - z_2| = k \Rightarrow$ Circle, if $k \neq 1$ or Perpendicular bisector, if $k = 1$

(iii) The equation of a circle described on the line segment joining z_1 and $_1$ as diameter is $(z - z_1)(z - z_2) + (z - z_2)(z - z_1) = 0$

(iv) If z_1, and z_2 are the fixed complex numbers, then the locus of a point z satisfying arg $[(z - z_1)/(z - z_2)] = \pm \pi / 2$ is a circle having z_1 and z_2 at the end points of a diameter.

Conic in Complex plane

(i) Let z_1 and z2 be two fixed points, and k be a positive real number.

If $k > |z_1 - z_2|$, then $|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of the major axis is k.

(ii) Let z_1 and z2 be two fixed points and k be a positive real number.

If $k \neq |z_1 - z_2|$, then $|z - z_1| - |z - z_2| = k$ represents hyperbola with foci at $A(z_1)$ and $B(z_2)$.

Important Points to be Remembered

- $\sqrt{-a} \times \sqrt{-b} \neq \sqrt{ab}$

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is possible only, if both a and b are non-negative.

So, $i^2 = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

- is neither positive, zero nor negative.
- Argument of 0 is not defined.
- Argument of purely imaginary number is $\pi/2$
- Argument of purely real number is 0 or π.
- If $|z + 1/z| = a$ then the greatest value of $|z| = a + \sqrt{a^2} + 4/2$ and the least value of $|z| = -a + \sqrt{a^2} + 4/2$
- The value of $i^i = e^{-\pi 2}$
- The complex number do not possess the property of order, i.e., $x + iy <$ (or) $> c + id$ is not defined.
- The area of the triangle on the Argand plane formed by the complex numbers z, iz and z + iz is $1/2|z|^2$.
- (x) If ω_1 and ω_2 are the complex slope of two lines on the Argand plane, then the lines are

(a) perpendicular, if $\omega_1 + \omega_2 = 0$.
(b) parallel, if $\omega_1 = \omega_2$.