Miscellaneous Examples

Example 9 (Diet problem) A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

Solution Let x and y be the number of packets of food P and Q respectively. Obviously $x \geq 0, y \geq 0$. Mathematical formulation of the given problem is as follows:
Minimise $Z = 6x + 3y$ (vitamin A)

subject to the constraints

$12x + 3y \geq 240$ (constraint on calcium), i.e. $4x + y \geq 80$ ... (1)

$4x + 20y \geq 460$ (constraint on iron), i.e. $x + 5y \geq 115$ ... (2)

$6x + 4y \leq 300$ (constraint on cholesterol), i.e. $3x + 2y \leq 150$ ... (3)

$x \geq 0, y \geq 0$ ... (4)

Let us graph the inequalities (1) to (4).

The feasible region (shaded) determined by the constraints (1) to (4) is shown in Fig 12.10 and note that it is bounded.

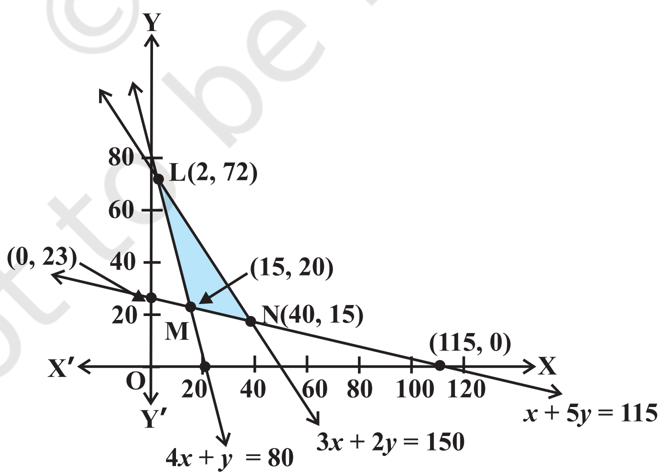

Fig 12.10

The coordinates of the corner points L, M and N are (2, 72), (15, 20) and (40, 15) respectively. Let us evaluate Z at these points:

Corner Point	$Z = 6x + 3y$	
(2, 72)	228	
(15, 20)	**150** ←	Minimum
(40, 15)	285	

From the table, we find that Z is minimum at the point (15, 20). Hence, the amount of vitamin A under the constraints given in the problem will be minimum, if 15 packets of food P and 20 packets of food Q are used in the special diet. The minimum amount of vitamin A will be 150 units.

Example 10 **(Manufacturing problem)** A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines.

The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Items	Number of hours required on machines		
	I	II	III
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

Solution Let x and y be the number of items M and N respectively.
Total profit on the production = Rs $(600x + 400y)$
Mathematical formulation of the given problem is as follows:
Maximise $Z = 600x + 400y$
subject to the constraints:

$$x + 2y \leq 12 \text{ (constraint on Machine I)} \quad \dots (1)$$
$$2x + y \leq 12 \text{ (constraint on Machine II)} \quad \dots (2)$$
$$x + \frac{5}{4}y \geq 5 \text{ (constraint on Machine III)} \quad \dots (3)$$
$$x \geq 0, \ y \geq 0 \quad \dots (4)$$

Let us draw the graph of constraints (1) to (4). ABCDE is the feasible region (shaded) as shown in Fig 12.11 determined by the constraints (1) to (4). Observe that the feasible region is bounded, coordinates of the corner points A, B, C, D and E are (5, 0) (6, 0), (4, 4), (0, 6) and (0, 4) respectively.

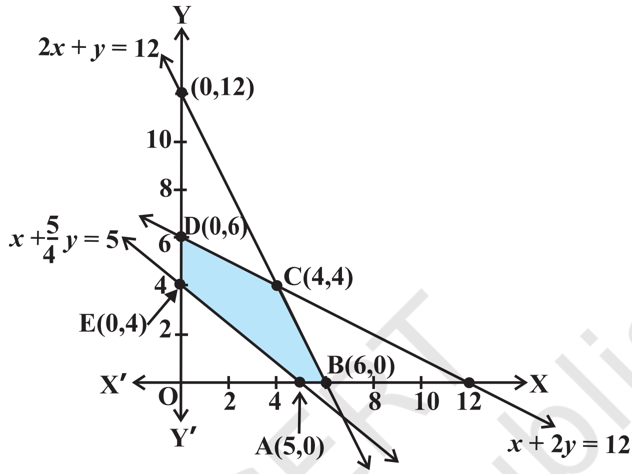

Fig 12.11

Let us evaluate Z = 600 x + 400 y at these corner points.

Corner point	Z = 600 x + 400 y	
(5, 0)	3000	
(6, 0)	3600	
(4, 4)	**4000** ←	Maximum
(0, 6)	2400	
(0, 4)	1600	

We see that the point (4, 4) is giving the maximum value of Z. Hence, the manufacturer has to produce 4 units of each item to get the maximum profit of Rs 4000.

Example 11 (Transportation problem) There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

From/To	Cost (in Rs)		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Solution The problem can be explained diagrammatically as follows (Fig 12.12):

Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then $(8 - x - y)$ units will be transported to depot at C (Why?)

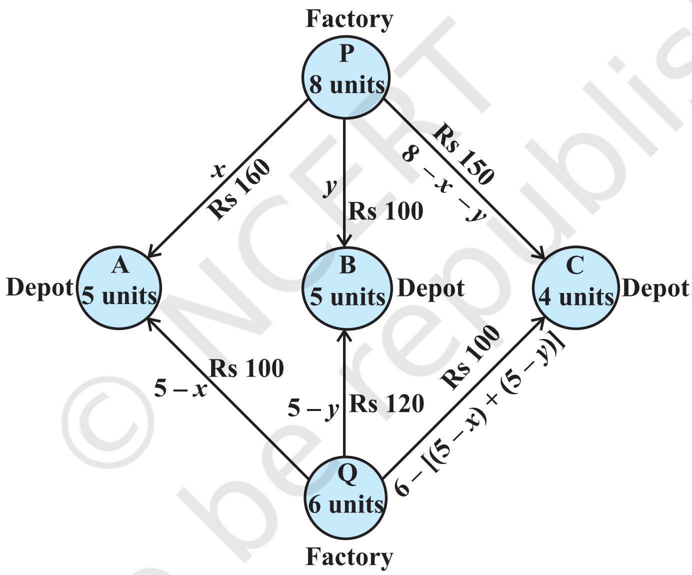

Fig 12.12

Hence, we have $x \geq 0, y \geq 0$ and $8 - x - y \geq 0$

i.e. $x \geq 0, y \geq 0$ and $x + y \leq 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5 - x)$ units need to be transported from the factory at Q. Obviously, $5 - x \geq 0$, i.e. $x \leq 5$.

Similarly, $(5 - y)$ and $6 - (5 - x + 5 - y) = x + y - 4$ units are to be transported from the factory at Q to the depots at B and C respectively.

Thus, $5 - y \geq 0$, $x + y - 4 \geq 0$

i.e. $y \leq 5$, $x + y \geq 4$

Total transportation cost Z is given by

$$Z = 160\,x + 100\,y + 150\,(8 - x - y) + 100\,(5 - x) + 120\,(5 - y) + 100\,(x + y - 4)$$
$$= 10\,(x - 7\,y + 190)$$

Therefore, the problem reduces to

Minimise $Z = 10\,(x - 7y + 190)$

subject to the constraints:

$x \geq 0, y \geq 0$ ... (1)

$x + y \leq 8$ ... (2)

$x \leq 5$ ... (3)

$y \leq 5$ ... (4)

and $x + y \geq 4$ ... (5)

The shaded region ABCDEF represented by the constraints (1) to (5) is the feasible region (Fig 12.13).

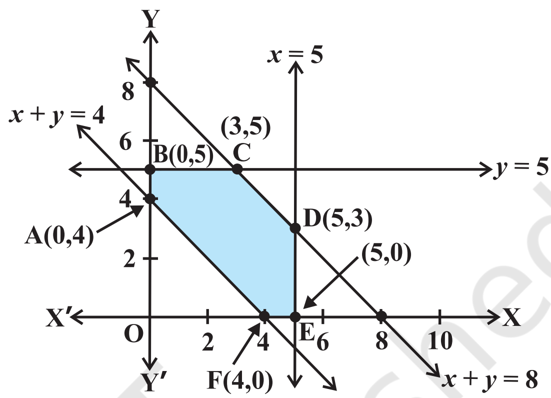

Fig 12.13

Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0). Let us evaluate Z at these points.

Corner Point	$Z = 10\,(x - 7\,y + 190)$	
(0, 4)	1620	
(0, 5)	1550 ←	Minimum
(3, 5)	1580	
(5, 3)	1740	
(5, 0)	1950	
(4, 0)	1940	

From the table, we see that the minimum value of Z is 1550 at the point (0, 5).

Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e., Rs 1550.

Miscellaneous Exercise on Chapter 12

1. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

2. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

3. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	Vitamin A	Vitamin B	Vitamin C
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

4. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Types of Toys	Machines		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

5. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

6. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per quintal (in Rs)		
From/To	**A**	**B**
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

7. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From / To	**A**	**B**
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

8. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine.

If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	**Brand Q**
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

9. Refer to Question 8. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

10. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?

Summary

- A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum) of a linear function of several variables (called **objective function)** subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear **constraints**). Variables are sometimes called **decision variables** and are **non-negative.**
- A few important linear programming problems are:
  (i) Diet problems
  (ii) Manufacturing problems
  (iii) Transportation problems
- The common region determined by all the constraints including the non-negative constraints $x \geq 0, y \geq 0$ of a linear programming problem is called the **feasible region** (or **solution region**) for the problem.
- Points within and on the boundary of the feasible region represent **feasible solutions** of the constraints.
  Any point outside the feasible region is an **infeasible solution.**

- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**
- The following Theorems are fundamental in solving linear programming problems:

  Theorem 1 Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

  Theorem 2 Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is **bounded**, then the objective function Z has both a **maximum** and a **minimum** value on R and each of these occurs at a corner point (vertex) of R.
- If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R.
- **Corner point method** for solving a linear programming problem. The method comprises of the following steps:
  (i) Find the feasible region of the linear programming problem and determine its corner points (vertices).
  (ii) Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.
  (iii) If the feasible region is bounded, M and m respectively are the maximum and minimum values of the objective function.

  If the feasible region is unbounded, then
  (i) M is the maximum value of the objective function, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.
  (ii) m is the minimum value of the objective function, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.
- If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

Historical Note

In the World War II, when the war operations had to be planned to economise expenditure, maximise damage to the enemy, linear programming problems came to the forefront.

The first problem in linear programming was formulated in 1941 by the Russian mathematician, L. Kantorovich and the American economist, F. L. Hitchcock, both of whom worked at it independently of each other. This was the well known *transportation problem*. In 1945, an English economist, G.Stigler, described yet another linear programming problem – that of determining an *optimal diet*.

In 1947, the American economist, G. B. Dantzig suggested an efficient method known as the simplex method which is an iterative procedure to solve any linear programming problem in a finite number of steps.

L. Katorovich and American mathematical economist, T. C. Koopmans were awarded the nobel prize in the year 1975 in economics for their pioneering work in linear programming. With the advent of computers and the necessary softwares, it has become possible to apply linear programming model to increasingly complex problems in many areas.