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Chapter 7

PERMUTATIONS AND COMBINATIONS

❖*Every body of discovery is mathematical in form because there is no other guidance we can have – DARWIN*❖

7.1 Introduction

Jacob Bernoulli (1654-1705)

Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition. Some how, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of these techniques.

7.2 Fundamental Principle of Counting

Let us consider the following problem. Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt.

Let us name the three pants as P_1, P_2, P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the Fig. 7.1.

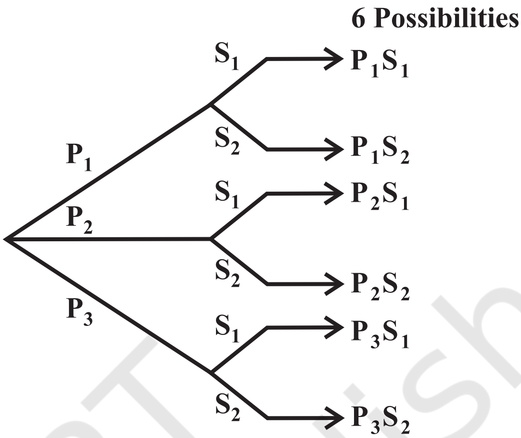

Fig 7.1

Let us consider another problem of the same type.

Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are $6 \times 2 = 12$ different ways in which, Sabnam can carry these items to school. If we name the 2 school bags as B_1, B_2, the three tiffin boxes as T_1, T_2, T_3 and the two water bottles as W_1, W_2, these possibilities can be illustrated in the Fig. 7.2.

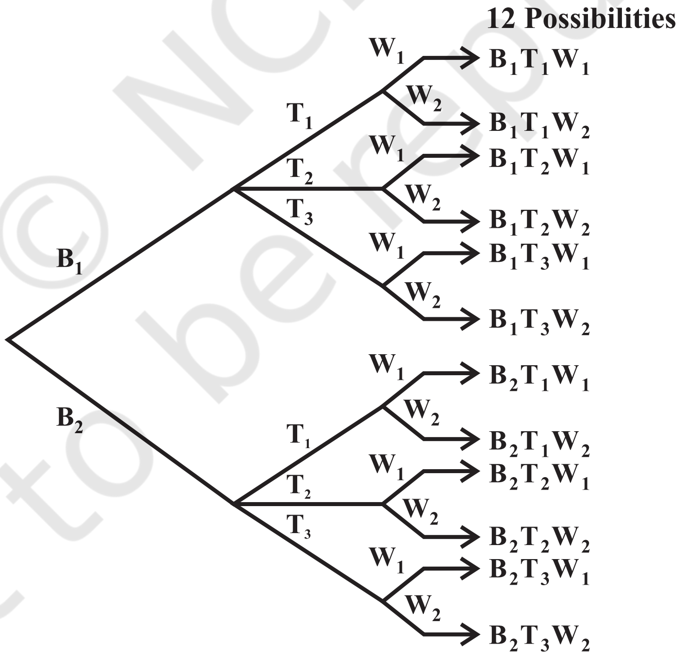

Fig 7.2

In fact, the problems of the above types are solved by applying the following principle known as the *fundamental principle of counting*, or, simply, the *multiplication principle*, which states that

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m×n."

The above principle can be generalised for any finite number of events. For example, for 3 events, the principle is as follows:

'If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to 'the events in the given order is $m \times n \times p$."

In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a pant
(ii) the event of choosing a shirt.

In the second problem, the required number of ways was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a school bag
(ii) the event of choosing a tiffin box
(iii) the event of choosing a water bottle.

Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurence of the events in this chosen order.

Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in 4 vacant places □ □ □ □ by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R,O,S,E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

> **Note** If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $4 \times 4 \times 4 \times 4 = 256$.

Example 2 Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = $4 \times 3 = 12$.

Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

Example 4 Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 flags.

The number of ways is $5 \times 4 \times 3 = 60$.

Continuing the same way, we find that

The number of 4 flag signals $= 5 \times 4 \times 3 \times 2 = 120$

and the number of 5 flag signals $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals $= 20 + 60 + 120 + 120 = 320$.

EXERCISE 7.1

1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that

(i) repetition of the digits is allowed?

(ii) repetition of the digits is not allowed?

2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

5. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

6. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

7.3 Permutations

In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a *permutation of* 4 *different letters taken all at a time*. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words $= 6 \times 5 \times 4 = 120$ (by using multiplication principle).

If the repetition of the letters was allowed, the required number of words would be $6 \times 6 \times 6 = 216$.

Definition 1 A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

In the following sub-section, we shall obtain the formula needed to answer these questions immediately.

7.3.1 *Permutations when all the objects are distinct*

Theorem 1 The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by ${}^nP_r$.

Proof There will be as many permutations as there are ways of filling in r vacant places □ □ □ . . . □ by

$\leftarrow$ r vacant places $\rightarrow$

the n objects. The first place can be filled in n ways; following which, the second place can be filled in $(n-1)$ ways, following which the third place can be filled in $(n-2)$ ways,..., the rth place can be filled in $(n-(r-1))$ ways. Therefore, the number of ways of filling in r vacant places in succession is $n(n-1)(n-2)\ldots(n-(r-1))$ or $n(n-1)(n-2)\ldots(n-r+1)$

This expression for ${}^nP_r$ is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol $n!$ (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually $n!$ means.

7.3.2 ***Factorial notation*** The notation $n!$ represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as $n!$. We read this symbol as 'n factorial'. Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n!$

$$1 = 1!$$
$$1 \times 2 = 2!$$
$$1 \times 2 \times 3 = 3!$$
$$1 \times 2 \times 3 \times 4 = 4! \text{ and so on.}$$

We define $0! = 1$

We can write $5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2!$
$= 5 \times 4 \times 3 \times 2 \times 1!$

Clearly, for a natural number n

$$n! = n(n-1)!$$
$$= n(n-1)(n-2)! \quad [\text{provided } (n \geq 2)]$$
$$= n(n-1)(n-2)(n-3)! \quad [\text{provided } (n \geq 3)]$$

and so on.

Example 5 Evaluate (i) 5 ! (ii) 7 ! (iii) 7 ! – 5!

Solution (i) $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

and (iii) $7! - 5! = 5040 - 120 = 4920$.

Example 6 Compute (i) $\frac{7!}{5!}$ (ii) $\frac{12!}{(10!)(2!)}$

Solution (i) We have $\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$

and (ii) $\frac{12!}{(10!)(2!)} = \frac{12 \times 11 \times (10!)}{(10!) \times (2)} = 6 \times 11 = 66$.

Example 7 Evaluate $\frac{n!}{r!(n-r)!}$, when $n = 5$, $r = 2$.

Solution We have to evaluate $\frac{5!}{2!(5-2)!}$ (since $n = 5$, $r = 2$)

We have $\frac{5!}{2!(5-2)!} = \frac{5!}{2! \times 3!} = \frac{5 \times 4}{2} = 10$.

Example 8 If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, find x.

Solution We have $\frac{1}{8!} + \frac{1}{9 \times 8!} = \frac{x}{10 \times 9 \times 8!}$

Therefore $1 + \frac{1}{9} = \frac{x}{10 \times 9}$ or $\frac{10}{9} = \frac{x}{10 \times 9}$

So $x = 100$.

EXERCISE 7.2

1. Evaluate

(i) 8 ! (ii) 4 ! – 3 !

2. Is 3 ! + 4 ! = 7 ! ? **3.** Compute $\frac{8!}{6! \times 2!}$ **4.** If $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$, find x

5. Evaluate $\frac{n!}{(n-r)!}$, when

(i) $n = 6$, $r = 2$ (ii) $n = 9$, $r = 5$.

7.3.3 *Derivation of the formula for* ^nP_r

$$^nP_r = \frac{n!}{(n-r)!},\ 0 \le r \le n$$

Let us now go back to the stage where we had determined the following formula:

$$^nP_r = n\ (n-1)\ (n-2)\ .\ .\ .\ \ (n-r+1)$$

Multiplying numerator and denomirator by $(n-r)\ (n-r-1)\ .\ .\ .\ 3 \times 2 \times 1$, we get

$$^nP_r = \frac{n(n-1)\,(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots3\times2\times1}{(n-r)(n-r-1)\ldots3\times2\times1} = \frac{n!}{(n-r)!},$$

Thus $$^nP_r = \frac{n!}{(n-r)!},\ \text{where } 0 < r \le n$$

This is a much more convenient expression for ^nP_r than the previous one.

In particular, when $r = n$, $^nP_n = \frac{n!}{0!} = n!$

Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we know that there is only one way of doing so. Thus, we can have

$$^nP_0 = 1 = \frac{n!}{n!} = \frac{n!}{(n-0)!} \qquad \ldots (1)$$

Therefore, the formula (1) is applicable for $r = 0$ also.

Thus $$^nP_r = \frac{n!}{(n-r)!},\ 0 \le r \le n\ .$$