12. Find the angle between the planes whose vector equations are

$\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}-3\hat{j}+5\hat{k})=3$.

13. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x+5y+6z+30=0$ and $3x-y-10z+4=0$

(b) $2x+y+3z-2=0$ and $x-2y+5=0$

(c) $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$

(d) $2x-y+3z-1=0$ and $2x-y+3z+3=0$

(e) $4x+8y+z-8=0$ and $y+z-4=0$

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

	Point	Plane
(a)	$(0, 0, 0)$	$3x-4y+12z=3$
(b)	$(3, -2, 1)$	$2x-y+2z+3=0$
(c)	$(2, 3, -5)$	$x+2y-2z=9$
(d)	$(-6, 0, 0)$	$2x-3y+6z-2=0$

Miscellaneous Examples

Example 26 A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$$

Solution A cube is a rectangular parallelopiped having equal length, breadth and height. Let OADBFEGC be the cube with each side of length a units. (Fig 11.21)

The four diagonals are OE, AF, BG and CD.

The direction cosines of the diagonal OE which is the line joining two points O and E are

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}},\ \frac{a-0}{\sqrt{a^2+a^2+a^2}},\ \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

i.e., $\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}$

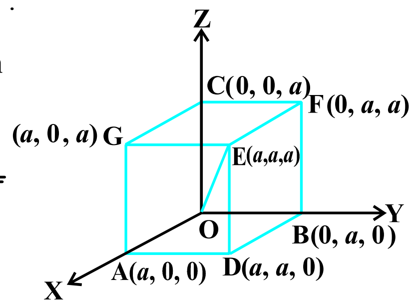

Fig 11.21

Similarly, the direction cosines of AF, BG and CD are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$, respectively.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OE, AF, BG, CD, respectively. Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l+m+n); \cos\beta = \frac{1}{\sqrt{3}}(-l+m+n);$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l-m+n); \cos\delta = \frac{1}{\sqrt{3}}(l+m-n) \quad \text{(Why?)}$$

Squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\left[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2\right]$$

$$= \frac{1}{3}\left[4(l^2+m^2+n^2)\right] = \frac{4}{3} \quad (\text{as } l^2+m^2+n^2 = 1)$$

Example 27 Find the equation of the plane that contains the point $(1, -1, 2)$ and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.

Solution The equation of the plane containing the given point is

$$A(x-1) + B(y+1) + C(z-2) = 0 \quad \dots (1)$$

Applying the condition of perpendicularly to the plane given in (1) with the planes

$$2x + 3y - 2z = 5 \text{ and } x + 2y - 3z = 8, \text{ we have}$$

$$2A + 3B - 2C = 0 \text{ and } A + 2B - 3C = 0$$

Solving these equations, we find $A = -5C$ and $B = 4C$. Hence, the required equation is

$$-5C(x-1) + 4C(y+1) + C(z-2) = 0$$

i.e. $\quad 5x - 4y - z = 7$

Example 28 Find the distance between the point P(6, 5, 9) and the plane determined by the points A(3, –1, 2), B(5, 2, 4) and C(–1, –1, 6).

Solution Let A, B, C be the three points in the plane. D is the foot of the perpendicular drawn from a point P to the plane. PD is the required distance to be determined, which is the projection of $\overrightarrow{AP}$ on $\overrightarrow{AB} \times \overrightarrow{AC}$.

Hence, PD = the dot product of $\overrightarrow{AP}$ with the unit vector along $\overrightarrow{AB} \times \overrightarrow{AC}$.

So $$\overrightarrow{AP} = 3\hat{i} + 6\hat{j} + 7\hat{k}$$

and $$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 12\hat{i} - 16\hat{j} + 12\hat{k}$$

Unit vector along $$\overrightarrow{AB} \times \overrightarrow{AC} = \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

Hence $$PD = (3\hat{i} + 6\hat{j} + 7\hat{k}) \cdot \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

$$= \frac{3\sqrt{34}}{17}$$

Alternatively, find the equation of the plane passing through A, B and C and then compute the distance of the point P from the plane.

Example 29 Show that the lines

$$\frac{x - a + d}{\alpha - \delta} = \frac{y - a}{\alpha} = \frac{z - a - d}{\alpha + \delta}$$

and $$\frac{x - b + c}{\beta - \gamma} = \frac{y - b}{\beta} = \frac{z - b - c}{\beta + \gamma}$$ are coplanar.

Solution

Here

$x_1 = a - d$ $\quad$ $x_2 = b - c$

$y_1 = a$ $\quad$ $y_2 = b$

$z_1 = a + d$ $\quad$ $z_2 = b + c$

$a_1 = \alpha - \delta$ $\quad$ $a_2 = \beta - \gamma$

$b_1 = \alpha$ $\quad$ $b_2 = \beta$

$c_1 = \alpha + \delta$ $\quad$ $c_2 = \beta + \gamma$

Now consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b - c - a + d & b - a & b + c - a - d \\ \alpha - \delta & \alpha & \alpha + \delta \\ \beta - \gamma & \beta & \beta + \gamma \end{vmatrix}$$

Adding third column to the first column, we get

$$2\begin{vmatrix} b-a & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

Since the first and second columns are identical. Hence, the given two lines are coplanar.

Example 30 Find the coordinates of the point where the line through the points A (3, 4, 1) and B(5, 1, 6) crosses the XY-plane.

Solution The vector equation of the line through the points A and B is

$$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda\,[\,(5-3)\hat{i} + (1-4)\hat{j} + (6-1)\hat{k}\,]$$

i.e. $$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda\,(2\hat{i} - 3\hat{j} + 5\hat{k}) \qquad \ldots (1)$$

Let P be the point where the line AB crosses the XY-plane. Then the position vector of the point P is of the form $x\hat{i} + y\hat{j}$.

This point must satisfy the equation (1). (Why ?)

i.e. $$x\hat{i} + y\hat{j} = (3+2\lambda)\hat{i} + (4-3\lambda)\hat{j} + (1+5\lambda)\hat{k}$$

Equating the like coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$x = 3 + 2\lambda$$
$$y = 4 - 3\lambda$$
$$0 = 1 + 5\lambda$$

Solving the above equations, we get

$$x = \frac{13}{5} \text{ and } y = \frac{23}{5}$$

Hence, the coordinates of the required point are $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

Miscellaneous Exercise on Chapter 11

1. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, – 1), (4, 3, – 1).
2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $m_1 n_2 - m_2 n_1,\ n_1 l_2 - n_2 l_1,\ l_1 m_2 - l_2 m_1$

3. Find the angle between the lines whose direction ratios are a, b, c and $b - c, c - a, a - b$.

4. Find the equation of a line parallel to x-axis and passing through the origin.

5. If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (−4, 3, −6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

6. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

7. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.

8. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$.

9. Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$.

10. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4,1) crosses the YZ-plane.

11. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX-plane.

12. Find the coordinates of the point where the line through (3, −4, −5) and (2, −3, 1) crosses the plane $2x + y + z = 7$.

13. Find the equation of the plane passing through the point (−1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

14. If the points (1, 1, p) and (−3, 0, 1) be equidistant from the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, then find the value of p.

15. Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis.

16. If O be the origin and the coordinates of P be (1, 2, −3), then find the equation of the plane passing through P and perpendicular to OP.

17. Find the equation of the plane which contains the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$, $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$.

18. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$.

19. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}\cdot(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}\cdot(3\hat{i} + \hat{j} + \hat{k}) = 6$.

20. Find the vector equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

21. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Choose the correct answer in Exercises 22 and 23.

22. Distance between the two planes: $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is

(A) 2 units (B) 4 units (C) 8 units (D) $\frac{2}{\sqrt{29}}$ units

23. The planes: $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

(A) Perpendicular (B) Parallel

(C) intersect y-axis (D) passes through $\left(0, 0, \frac{5}{4}\right)$

Summary

- **Direction cosines of a line** are the cosines of the angles made by the line with the positive directions of the coordinate axes.
- If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.
- Direction cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- **Direction ratios of a line** are the numbers which are proportional to the direction cosines of a line.
- If l, m, n are the direction cosines and a, b, c are the direction ratios of a line

then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}\ ;\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}\ ;\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- **Skew lines** are lines in space which are neither parallel nor intersecting. They lie in different planes.
- **Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the acute angle between the two lines; then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$.
- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- The vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$.
- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.
- If θ is the acute angle between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$, then

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1|\ |\vec{b}_2|} \right|$$

- If $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$.

- Shortest distance between two skew lines is the line segment perpendicular to both the lines.
- Shortest distance between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is

$$\left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

- Shortest distance between the lines: $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and

$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

- Distance between parallel lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is

$$\left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

- In the vector form, equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is $\vec{r} \cdot \hat{n} = d$.
- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}) . \vec{N} = 0$.
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is

$$A (x - x_1) + B (y - y_1) + C (z - z_1) = 0$$

- Equation of a plane passing through three non collinear points (x_1, y_1, z_1),

(x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Vector equation of a plane that contains three non collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$
- Equation of a plane that cuts the coordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- Vector equation of a plane that passes through the intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$, where λ is any nonzero constant.
- Cartesian equation of a plane that passes through the intersection of two given planes $A_1 x + B_1 y + C_1 z + D_1 = 0$ and $A_2 x + B_2 y + C_2 z + D_2 = 0$ is $(A_1 x + B_1 y + C_1 z + D_1) + \lambda(A_2 x + B_2 y + C_2 z + D_2) = 0$.
- Two lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ are coplanar if

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$$

- In the cartesian form above lines passing through the points $A(x_1, y_1, z_1)$ and $B\ (x_2, y_2, z_2)$

$$= \frac{y \text{ ï } y_2}{b_2} = \frac{z \text{ ï } z_2}{C_2} \text{ are coplanar if } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

- In the vector form, if θ is the angle between the two planes, $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then $\theta = \cos^{ï1} \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$.
- The angle ϕ between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \hat{n} = d$ is

$$\sin \phi = \left| \frac{\vec{b} \cdot \hat{n}}{|\vec{b}|\,|\hat{n}|} \right|$$

- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2 x + B_2 y + C_2 z + D_2 = 0$ is given by

$$\cos \theta = \left| \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

- The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r} \cdot \hat{n} = d$ is $|d - \vec{a} \cdot \hat{n}|$
- The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is

$$\left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$