

Clearly $\Delta_1 = -\Delta$

Hence, Property 2 is verified.

Property 3 If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

Proof If we interchange the identical rows (or columns) of the determinant Δ , then Δ does not change. However, by Property 2, it follows that Δ has changed its sign

Therefore $\Delta = -\Delta$

or $\Delta = 0$

Let us verify the above property by an example.

Example 8 Evaluate $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

Solution Expanding along first row, we get

$$\begin{aligned} \Delta &= 3(6 - 6) - 2(6 - 9) + 3(4 - 6) \\ &= 0 - 2(-3) + 3(-2) = 6 - 6 = 0 \end{aligned}$$

Here R_1 and R_3 are identical.

Property 4 If each element of a row (or a column) of a determinant is multiplied by a constant k , then its value gets multiplied by k .

Verification Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and Δ_1 be the determinant obtained by multiplying the elements of the first row by k . Then

$$\Delta_1 = \begin{vmatrix} k a_1 & k b_1 & k c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along first row, we get

$$\begin{aligned} \Delta_1 &= k a_1 (b_2 c_3 - b_3 c_2) - k b_1 (a_2 c_3 - c_2 a_3) + k c_1 (a_2 b_3 - b_2 a_3) \\ &= k [a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - c_2 a_3) + c_1 (a_2 b_3 - b_2 a_3)] \\ &= k \Delta \end{aligned}$$

Hence
$$\begin{vmatrix} k a_1 & k b_1 & k c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Remarks

- (i) By this property, we can take out any common factor from any one row or any one column of a given determinant.
- (ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero. For example

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ k a_1 & k a_2 & k a_3 \end{vmatrix} = 0 \text{ (rows } R_1 \text{ and } R_2 \text{ are proportional)}$$

Example 9 Evaluate
$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Solution Note that
$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$$

(Using Properties 3 and 4)

Property 5 If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

For example,
$$\begin{vmatrix} a_1 + \lambda_1 & a_2 + \lambda_2 & a_3 + \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Verification L.H.S. =
$$\begin{vmatrix} a_1 + \lambda_1 & a_2 + \lambda_2 & a_3 + \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expanding the determinants along the first row, we get

$$\begin{aligned}\Delta &= (a_1 + \lambda_1) (b_2 c_3 - c_2 b_3) - (a_2 + \lambda_2) (b_1 c_3 - b_3 c_1) \\ &\quad + (a_3 + \lambda_3) (b_1 c_2 - b_2 c_1) \\ &= a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1) \\ &\quad + \lambda_1 (b_2 c_3 - c_2 b_3) - \lambda_2 (b_1 c_3 - b_3 c_1) + \lambda_3 (b_1 c_2 - b_2 c_1) \\ &\hspace{15em} \text{(by rearranging terms)}\end{aligned}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{R.H.S.}$$

Similarly, we may verify Property 5 for other rows or columns.

Example 10 Show that $\begin{vmatrix} a & b & c \\ a + 2x & b + 2y & c + 2z \\ x & y & z \end{vmatrix} = 0$

Solution We have $\begin{vmatrix} a & b & c \\ a + 2x & b + 2y & c + 2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

(by Property 5)

$= 0 + 0 = 0$ (Using Property 3 and Property 4)

Property 6 If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$.

Verification

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a_1 + kc_1 & a_2 + kc_2 & a_3 + kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

where Δ_1 is obtained by the operation $R_1 \rightarrow R_1 + kR_3$.

Here, we have multiplied the elements of the third row (R_3) by a constant k and added them to the corresponding elements of the first row (R_1).

Symbolically, we write this operation as $R_1 \rightarrow R_1 + kR_3$.

Now, again

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & kc_2 & kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (\text{Using Property 5})$$

$$= \Delta + 0 \quad (\text{since } R_1 \text{ and } R_3 \text{ are proportional})$$

Hence $\Delta = \Delta_1$

Remarks

- (i) If Δ_1 is the determinant obtained by applying $R_i \rightarrow kR_i$ or $C_i \rightarrow kC_i$ to the determinant Δ , then $\Delta_1 = k\Delta$.
- (ii) If more than one operation like $R_i \rightarrow R_i + kR_j$ is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operations.

Example 11 Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$.

Solution Applying operations $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ to the given determinant Δ , we have

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying $R_3 \rightarrow R_3 - 3R_2$, we get

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1 , we obtain

$$\Delta = a \begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0 + 0$$

$$= a(a^2 - 0) = a(a^2) = a^3$$

Example 12 Without expanding, prove that

$$\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solution Applying $R_1 \rightarrow R_1 + R_2$ to Δ , we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

Since the elements of R_1 and R_3 are proportional, $\Delta = 0$.

Example 13 Evaluate

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Solution Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

Taking factors $(b-a)$ and $(c-a)$ common from R_2 and R_3 , respectively, we get

$$\begin{aligned} \Delta &= (b-a)(c-a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix} \\ &= (b-a)(c-a)[(-b+c)] \text{ (Expanding along first column)} \\ &= (a-b)(b-c)(c-a) \end{aligned}$$

Example 14 Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Solution Let $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \rightarrow R_1 - R_2 - R_3$ to Δ , we get

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expanding along R_1 , we obtain

$$\begin{aligned} \Delta &= 0 \begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c) \begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b) \begin{vmatrix} b & c+a \\ c & c \end{vmatrix} \\ &= 2c(ab + b^2 - bc) - 2b(bc - c^2 - ac) \\ &= 2abc + 2cb^2 - 2bc^2 - 2b^2c + 2bc^2 + 2abc \\ &= 4abc \end{aligned}$$

Example 15 If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then

show that $1 + xyz = 0$

Solution We have

$$\begin{aligned} \Delta &= \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} \\ &= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \quad (\text{Using Property 5}) \\ &= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (\text{Using } C_3 \leftrightarrow C_2 \text{ and then } C_1 \leftrightarrow C_2) \\ &= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} (1+xyz) \end{aligned}$$

$$= (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \quad (\text{Using } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1)$$

Taking out common factor $(y-x)$ from R_2 and $(z-x)$ from R_3 , we get

$$\Delta = (1+xyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

$$= (1+xyz)(y-x)(z-x)(z-y) \quad (\text{on expanding along } C_1)$$

Since $\Delta = 0$ and x, y, z are all different, i.e., $x-y \neq 0$, $y-z \neq 0$, $z-x \neq 0$, we get $1+xyz = 0$

Example 16 Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = abc + bc + ca + ab$$

Solution Taking out factors a, b, c common from R_1, R_2 and R_3 , we get

$$\text{L.H.S.} = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$


Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have

$$\Delta = abc \begin{vmatrix} 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix}$$

$$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix}$$

Now applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$, we get

$$\begin{aligned} \Delta &= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix} \\ &= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) [1(1-0)] \\ &= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = abc + bc + ca + ab = \text{R.H.S.} \end{aligned}$$

 **Note** Alternately try by applying $C_1 \rightarrow C_1 - C_2$ and $C_3 \rightarrow C_3 - C_2$, then apply $C_1 \rightarrow C_1 - a C_3$.

EXERCISE 4.2

Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

2. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

4. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

5. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

$$6. \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

$$7. \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2 b^2 c^2$$

By using properties of determinants, in Exercises 8 to 14, show that:

$$8. (i) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$(ii) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

$$9. \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

$$10. (i) \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

$$(ii) \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

$$11. (i) \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$(ii) \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

$$12. \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

$$13. \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

$$14. \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Choose the correct answer in Exercises 15 and 16.

15. Let A be a square matrix of order 3×3 , then $|kA|$ is equal to
 (A) $k|A|$ (B) $k^2|A|$ (C) $k^3|A|$ (D) $3k|A|$
16. Which of the following is correct
 (A) Determinant is a square matrix.
 (B) Determinant is a number associated to a matrix.
 (C) Determinant is a number associated to a square matrix.
 (D) None of these

4.4 Area of a Triangle

In earlier classes, we have studied that the area of a triangle whose vertices are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , is given by the expression $\frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]$. Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \dots (1)$$

Remarks

- (i) Since area is a positive quantity, we always take the absolute value of the determinant in (1).