22. Prove that is divisible by *a* + *b* + *c* and find the quotient.

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

Solution:

$$\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

Now, [Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$]

$$\Delta = \begin{vmatrix} bc-a^2-ca+b^2 & ca-b^2-ab+c^2 & ab-c^2 \\ ca-b^2-ab+c^2 & ab-c^2-bc+a^2 & bc-a^2 \\ ab-c^2-bc+a^2 & bc-a^2-ca+b^2 & ca-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab-c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc-a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca-b^2 \end{vmatrix}$$

Next,

[Taking $(a+b+c)$ common from C_1 and C_2 each]

$$\Delta = (a+b+c)^2 \begin{vmatrix} b-a & c-b & ab-c^2 \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

Then,

[Applying $R_1 \to R_1 + R_2 + R_3$]

$$\Delta = (a+b+c)^2 \begin{vmatrix} 0 & 0 & ab+bc+ca-(a^2+b^2+c^2) \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

Lastly,

[Expanding along R_1]

$$\Delta = (a+b+c)^2[ab+bc+ca-(a^2+b^2+c^2)][(c-b)(b-a)-(a-c)^2]$$

$$= (a+b+c)^2(ab+bc+ca-a^2-b^2-c^2) \times (bc-ac-b^2+ab-a^2-c^2+2ac)$$

$$= (a+b+c)[(a+b+c)\,(a^2+b^2+c^2-ab-bc-ca)^2]$$

Therefore, given determinant is divisible by $(a+b+c)$ and quotient is

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2$$

$$\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

Now, [Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$]

$$\Delta = \begin{vmatrix} bc-a^2-ca+b^2 & ca-b^2-ab+c^2 & ab-c^2 \\ ca-b^2-ab+c^2 & ab-c^2-bc+a^2 & bc-a^2 \\ ab-c^2-bc+a^2 & bc-a^2-ca+b^2 & ca-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab-c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc-a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca-b^2 \end{vmatrix}$$

Next,

[Taking $(a+b+c)$ common from C_1 and C_2 each]

$$\Delta = (a+b+c)^2 \begin{vmatrix} b-a & c-b & ab-c^2 \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

Then,

[Applying $R_1 \to R_1 + R_2 + R_3$]

$$\Delta = (a+b+c)^2 \begin{vmatrix} 0 & 0 & ab+bc+ca-(a^2+b^2+c^2) \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

Lastly,

[Expanding along R_1]

$$\Delta = (a+b+c)^2[ab+bc+ca-(a^2+b^2+c^2)][(c-b)(b-a)-(a-c)^2]$$

$$= (a+b+c)^2(ab+bc+ca-a^2-b^2-c^2) \times (bc-ac-b^2+ab-a^2-c^2+2ac)$$

$$= (a+b+c)[(a+b+c)\,(a^2+b^2+c^2-ab-bc-ca)^2]$$

Therefore, given determinant is divisible by $(a+b+c)$ and quotient is

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2$$