1 What is Determinants?

Determinant is a real number that is given only for square matrices. They have many uses in mathematical field as well as in real life engineering situations. For example, they are used to check for the existance of the solutions of a system of equations. In 2-D geometry Determinant can be thought of as area of a Parallelogram which is formed from the columns of a matrix. Notations,

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

here Δ is the determinat of 2 x 2 matrix.

2 Minor and Cofactor of Matrix

Let us consider a 3 x 3 matrix to get minor and cofactor definition,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

here a_{ij} is an element from i^{th} row and j^{th} column.
Minor of element a_{ij} is defined as the value of the determinant obtained by eleminating the i^{th} row and j^{th} column of Δ. We donote the minor of a_{ij} by M_{ij}. For examples,

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a23a32$$

Now for cofactor we just have to assign a sign to the corresponding minor. Cofactor of a_{ij} is denoted by C_{ij}. And it is given by,

$$\text{Cofactor of } a_{ij} = C_{ij} = \underbrace{(-1)^{i+j}}_{\text{Sign}} \text{ Minor of } a_{ij}$$

For example,

$$\text{Cofactor of } a_{11} = C_{11} = \underbrace{(-1)^{i+j}}_{\text{Sign}} \text{ Minor of } a_{11} = (-1)^{1+1}M_{11} = M_{11}$$

The determinant of any general 3 x 3 matrix is given by,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Note: Determinant of a matrix can be written for any row or column of the matrix. For understanding purposes, we have written for only first row.

3 Examples of Determinants

Determinant of 1 x 1 matrix is simply that number itself. So,

$$\Delta = |a_{11}| = a_{11}$$

For 2 x 2 matrix,

$$\begin{aligned}\Delta &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{11}(-1)^{1+1}a_{22} + a_{12}(-1)^{1+2}a_{21} \\ &= a_{11}a_{22} - a_{12}a_{21}\end{aligned}$$

For 3 x 3 matrix,

$$\begin{aligned}\Delta &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}(-1)^{(1+1)}(a_{22}a_{33} - a23a32) + a_{12}(-1)^{(1+2)}(a_{21}a_{33} - a23a31) + a_{13}(-1)^{(1+3)}(a_{21}a_{32} - a22a31) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{1} - a_{12}a_{21}a_{33}\end{aligned}$$