5. Find the Taylor Series for $f\left(x\right)=\dfrac{7}{x^4}$ about $x=-3$.

Step 1
Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$\begin{aligned}
n=0: &\quad f\left(x\right)=\frac{7}{x^4}=7x^{-4}\\
n=1: &\quad f'\left(x\right)=-7\left(4\right)x^{-5}\\
n=2: &\quad f''\left(x\right)=7\left(4\right)\left(5\right)x^{-6}\\
n=3: &\quad f^{(3)}\left(x\right)=-7\left(4\right)\left(5\right)\left(6\right)x^{-7}\\
n=4: &\quad f^{(4)}\left(x\right)=7\left(4\right)\left(5\right)\left(6\right)\left(7\right)x^{-8}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n=4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula. In this case the only "simplification" we did was to multiply out the minus signs that came from the exponents upon doing the derivatives. That is a fairly common thing to do with these kinds of problems.

Step 2
It is now time to see if we can get a formula for the general term in the Taylor Series.

Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$\begin{aligned}
f^{(n)}\left(x\right)&=7\left(-1\right)^n\frac{\left(2\right)\left(3\right)}{\left(2\right)\left(3\right)}\left(4\right)\left(5\right)\left(6\right)\cdots\left(n+3\right)x^{-8}\\
&=7\left(-1\right)^n\frac{\left(2\right)\left(3\right)\left(4\right)\left(5\right)\left(6\right)\cdots\left(n+3\right)}{6}x^{-8}\\
&=\frac{7}{6}\left(-1\right)^n\left(n+3\right)!x^{-(n+4)} \qquad\qquad n=0,1,2,3,\ldots
\end{aligned}$$

This formula may have been a little trickier to get. We almost had a factorial in the derivatives but each one was missing the $\left(2\right)\left(3\right)$ part that would be needed to get the factorial to show up. Because that was all that was missing and it was missing in each of the derivatives we multiplied each derivative by

$\frac{(2)(3)}{(2)(3)}$ (*i.e.* a really fancy way of writing one…). We could then use the numerator of this to complete the factorial and the denominator was just left alone.

Also, as noted this formula works all the way back to $n = 0$. It is important to make sure that you check this formula to determine just how far back it will work. We will, on occasion, get formulas that will not work for the first couple of *n*'s and we need to know that before we start writing down the Taylor Series.

Step 3

Now, recall that we don't really want the general term at any *x*. We want the general term at $x = -3$. This is,

$$\begin{aligned} f^{(n)}(-3) &= \frac{7}{6}(-1)^n (n+3)!(-3)^{-(n+4)} \\ &= \frac{7(-1)^n (n+3)!}{6(-3)^{n+4}} \\ &= \frac{7(-1)^n (n+3)!}{6(-1)^{n+4}(3)^{n+4}} \\ &= \frac{7(n+3)!}{6(-1)^4(3)^{n+4}} \\ &= \frac{7(n+3)!}{6(3)^{n+4}} \qquad n = 1, 2, 3, \ldots \end{aligned}$$

We did a little simplification here so we could cancel out all the alternating signs that were present in the term.

Step 4

Okay, at this point we can formally write down the Taylor Series for this problem.

$$\frac{7}{x^4} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{7(n+3)!}{6(3)^{n+4}\, n!}(x+3)^n = \boxed{\sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}}(x+3)^n}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.