4. Find the Taylor Series for $f(x) = \ln(3+4x)$ about $x = 0$.

Step 1

Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$
\begin{aligned}
n=0: &\quad f(x) = \ln(3+4x) \\
n=1: &\quad f'(x) = \frac{4}{3+4x} = 4(3+4x)^{-1} \\
n=2: &\quad f''(x) = -4^2(3+4x)^{-2} \\
n=3: &\quad f^{(3)}(x) = 4^3(2)(3+4x)^{-3} \\
n=4: &\quad f^{(4)}(x) = -4^4(2)(3)(3+4x)^{-4} \\
n=5: &\quad f^{(5)}(x) = 4^5(2)(3)(4)(3+4x)^{-5}
\end{aligned}
$$

Remember that, in general, we're going to need to go out to at least $n = 4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula. In this case we "merged" all the 4's that came from the chain rule into a single term but left it as an exponent rather than get an actual value. This is not uncommon with these kinds of problems. The exponents we dropped down for the derivatives we left alone with the exception of dealing with the signs.

Step 2

It is now time to see if we can get a formula for the general term in the Taylor Series.

Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$\begin{aligned} f^{(0)}(x) &= \ln(3+4x) && n=0 \\ f^{(n)}(x) &= (-1)^{n+1} 4^n (n-1)!(3+4x)^{-n} && n=1,2,3,\ldots \end{aligned}$$

As noted this formula works all the way back to $n=1$ but clearly does not work for $n=0$. It is problems like this one that make it clear why we always need to check our proposed formula for the general solution to see just how far back it works.

Step 3

Now, recall that we don't really want the general term at any *x*. We want the general term at $x=0$. This is,

$$\begin{aligned} f^{(0)}(0) &= \ln(3) && n=0 \\ f^{(n)}(0) &= (-1)^{n+1} 4^n (n-1)!(3)^{-n} \\ &= (-1)^{n+1} 4^n (n-1)!\frac{1}{3^n} \\ &= (-1)^{n+1}\left(\frac{4}{3}\right)^n (n-1)! && n=1,2,3,\ldots \end{aligned}$$

We did a little simplification for the second one just to make it a little simpler.

Step 4

Okay, at this point we can formally write down the Taylor Series for this problem. However, before we actually do that recall that our general term formula did not work for $n=0$ and so we'll need to first strip that out of the series before we put the general formula in.

$$\begin{aligned} \ln(3+4x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n (n-1)!}{n!} x^n \\ &= \boxed{\ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n}{n} x^n} \end{aligned}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.