5. A rocket accelerates straight up by ejecting gas downwards. In a small time interval Δt, it ejects a gas of mass Δm at a relative speed u. Calculate KE of the entire system at $(t+\Delta t)$ and t and show that the device that ejects gas does work $= \frac{1}{2}\Delta m u^2$ in this time interval (neglect gravity).

Sol. Let mass of rocket at any time t = M

Velocity of rocket at any time t = v

Δm is the mass of gas ejected in time interval Δt

$(KE)_{z+\Delta t} = \frac{1}{2}(M-\Delta m)(v+\Delta v)^2 + \frac{1}{2}\Delta m(v-u)^2$

$= \frac{1}{2}\left[(M-\Delta m)\left(v^2+\Delta v^2+2v\Delta v\right)+\Delta m\left(v^2+u^2-2uv\right)\right]$

$$(KE)_{\tau+\Delta t} = \frac{1}{2}\begin{bmatrix} Mv^2+M\Delta v^2+2Mv\Delta v-\Delta mv^2 \\ -\Delta m\Delta v^2-2v\Delta m\Delta v+\Delta mv^2+ \\ \Delta mu^2-2uv\Delta m \end{bmatrix}$$

$(KE)_{t+\Delta t} = \frac{1}{2}Mv^2 + Mv\Delta v + \frac{1}{2}\Delta mu^2 - uv\Delta m$

[neglecting the very small terms $M\Delta v^2, \Delta m\Delta v^2, 2v\Delta m\Delta v$contains $\Delta v^2$and $\Delta m\Delta v$]

$(KE)_{\tau} = \frac{1}{2}Mv^2$

$(KE)_{t+\Delta t} - (KE)_t = \frac{1}{2}Mv^2 + Mv\Delta v + \frac{1}{2}\Delta mu^2 - uv\Delta m - \frac{1}{2}Mv^2$

$\Delta K = \frac{1}{2}\Delta mu^2 + v(M\Delta v - u\Delta m)$

By Newton's third law,

Reaction force on Rocket (upward) = Action force by burnt gases (downward)

$M\frac{dv}{dt} = \frac{dm}{dt}|u| (\because F = ma)$

Or $M\Delta v = \Delta mu \Rightarrow M\Delta v - u\Delta m = 0$

Substitute this value in (i)

$K = \frac{1}{2}u^2\Delta m$

By work energy theorem $\Delta(KE) = WD$

Or $W = \Delta K = \frac{1}{2}\Delta mu^2$.