8.2.1 *Binomial theorem for any positive integer n*,

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

Proof The proof is obtained by applying principle of mathematical induction.

Let the given statement be

$$P(n) : (a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

For $n = 1$, we have

$$P(1) : (a+b)^1 = {}^1C_0a^1 + {}^1C_1b^1 = a+b$$

Thus, P (1) is true.

Suppose P (k) is true for some positive integer k, i.e.

$$(a+b)^k = {}^kC_0a^k + {}^kC_1a^{k-1}b + {}^kC_2a^{k-2}b^2 + ...+ {}^kC_kb^k \quad ... (1)$$

We shall prove that P(k + 1) is also true, i.e.,

$$(a+b)^{k+1} = {}^{k+1}C_0\,a^{k+1} + {}^{k+1}C_1\,a^kb + {}^{k+1}C_2\,a^{k-1}b^2 + ...+ {}^{k+1}C_{k+1}\,b^{k+1}$$

Now, $(a+b)^{k+1} = (a+b)(a+b)^k$

$$= (a+b)({}^kC_0\,a^k + {}^kC_1a^{k-1}\,b + {}^kC_2\,a^{k-2}\,b^2 + ...+ {}^kC_{k-1}\,ab^{k-1} + {}^kC_k\,b^k)$$ [from (1)]

$$= {}^kC_0\,a^{k+1} + {}^kC_1\,a^kb + {}^kC_2a^{k-1}b^2 + ...+ {}^kC_{k-1}\,a^2b^{k-1} + {}^kC_k\,ab^k + {}^kC_0\,a^kb$$
$$+ {}^kC_1a^{k-1}b^2 + {}^kC_2a^{k-2}b^3 + ...+ {}^kC_{k-1}ab^k + {}^kC_kb^{k+1}$$ [by actual multiplication]

$$= {}^kC_0a^{k+1} + ({}^kC_1 + {}^kC_0)\,a^kb + ({}^kC_2 + {}^kC_1)a^{k-1}b^2 + ...$$
$$+ ({}^kC_k + {}^kC_{k-1})\,ab^k + {}^kC_kb^{k+1}$$ [grouping like terms]

$$= {}^{k+1}C_0a^{k+1} + {}^{k+1}C_1a^kb + {}^{k+1}C_2\,a^{k-1}b^2 + ...+ {}^{k+1}C_kab^k + {}^{k+1}C_{k+1}\,b^{k+1}$$

(by using ${}^{k+1}C_0 = 1$, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$ and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$)

Thus, it has been proved that P (k + 1) is true whenever P(k) is true. Therefore, by principle of mathematical induction, P(n) is true for every positive integer n.

We illustrate this theorem by expanding $(x+2)^6$:

$$(x+2)^6 = {}^6C_0x^6 + {}^6C_1x^5.2 + {}^6C_2x^42^2 + {}^6C_3x^3.2^3 + {}^6C_4x^2.2^4 + {}^6C_5x.2^5 + {}^6C_6.2^6.$$
$$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$$

Thus $(x+2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$.

Observations

1. The notation $\sum_{k=0}^{n} {}^{n}C_k\, a^{n-k} b^k$ stands for

${}^{n}C_0 a^n b^0 + {}^{n}C_1 a^{n-1} b^1 + ... + {}^{n}C_r a^{n-r} b^r + ... + {}^{n}C_n a^{n-n} b^n$, where $b^0 = 1 = a^{n-n}$.

Hence the theorem can also be stated as

$$(a+b)^n = \sum_{k=0}^{n} {}^{n}C_k\, a^{n-k} b^k .$$

2. The coefficients ${}^{n}C_r$ occuring in the binomial theorem are known as binomial coefficients.

3. There are $(n+1)$ terms in the expansion of $(a+b)^n$, i.e., one more than the index.

4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, $(n–1)$ in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term.

5. In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n + 0 = n$ in the first term, $(n – 1) + 1 = n$ in the second term and so on $0 + n = n$ in the last term. Thus, it can be seen that the sum of the indices of a and b is n in every term of the expansion.

8.2.2 *Some special cases* In the expansion of $(a + b)^n$,

(i) Taking $a = x$ and $b = – y$, we obtain

$$\begin{aligned}(x - y)^n &= [x + (-y)]^n \\ &= {}^{n}C_0 x^n + {}^{n}C_1 x^{n-1}(-y) + {}^{n}C_2 x^{n-2}(-y)^2 + {}^{n}C_3 x^{n-3}(-y)^3 + ... + {}^{n}C_n (-y)^n \\ &= {}^{n}C_0 x^n - {}^{n}C_1 x^{n-1} y + {}^{n}C_2 x^{n-2} y^2 - {}^{n}C_3 x^{n-3} y^3 + ... + (-1)^n\, {}^{n}C_n y^n\end{aligned}$$

Thus $(x–y)^n = {}^{n}C_0 x^n - {}^{n}C_1 x^{n-1} y + {}^{n}C_2 x^{n-2} y^2 + ... + (-1)^n\, {}^{n}C_n y^n$

Using this, we have

$$\begin{aligned}(x-2y)^5 &= {}^{5}C_0 x^5 - {}^{5}C_1 x^4 (2y) + {}^{5}C_2 x^3 (2y)^2 - {}^{5}C_3 x^2 (2y)^3 + {}^{5}C_4 x (2y)^4 - {}^{5}C_5 (2y)^5 \\ &= x^5 - 10x^4 y + 40x^3 y^2 - 80x^2 y^3 + 80xy^4 - 32y^5.\end{aligned}$$

(ii) Taking $a = 1$, $b = x$, we obtain

$$\begin{aligned}(1 + x)^n &= {}^{n}C_0 (1)^n + {}^{n}C_1 (1)^{n-1} x + {}^{n}C_2 (1)^{n-2} x^2 + ... + {}^{n}C_n x^n \\ &= {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + {}^{n}C_3 x^3 + ... + {}^{n}C_n x^n\end{aligned}$$

Thus $(1 + x)^n = {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + {}^{n}C_3 x^3 + ... + {}^{n}C_n x^n$

In particular, for $x = 1$, we have

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n.$$

(iii) Taking $a = 1$, $b = -x$, we obtain

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots + (-1)^n\, {}^nC_n x^n$$

In particular, for $x = 1$, we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots + (-1)^n\, {}^nC_n$$

Example 1 Expand $\left(x^2 + \frac{3}{x}\right)^4$, $x \neq 0$

Solution By using binomial theorem, we have

$$\left(x^2+\frac{3}{x}\right)^4 = {}^4C_0 (x^2)^4 + {}^4C_1 (x^2)^3 \left(\frac{3}{x}\right) + {}^4C_2 (x^2)^2 \left(\frac{3}{x}\right)^2 + {}^4C_3 (x^2) \left(\frac{3}{x}\right)^3 + {}^4C_4 \left(\frac{3}{x}\right)^4$$

$$= x^8 + 4.x^6 \cdot \frac{3}{x} + 6.x^4 \cdot \frac{9}{x^2} + 4.x^2.\frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Example 2 Compute $(98)^5$.

Solution We express 98 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem.

Write $98 = 100 - 2$

Therefore, $(98)^5 = (100 - 2)^5$

$$= {}^5C_0\, (100)^5 - {}^5C_1\, (100)^4.2 + {}^5C_2\, (100)^3 2^2$$

$$- {}^5C_3\, (100)^2\, (2)^3 + {}^5C_4\, (100)\, (2)^4 - {}^5C_5\, (2)^5$$

$$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000 \times 8 + 5 \times 100 \times 16 - 32$$

$$= 10040008000 - 1000800032 = 9039207968.$$

Example 3 Which is larger $(1.01)^{1000000}$ or 10,000?

Solution Splitting 1.01 and using binomial theorem to write the first few terms we have

$$(1.01)^{1000000} = (1 + 0.01)^{1000000}$$

$$= {}^{1000000}C_0 + {}^{1000000}C_1(0.01) + \text{other positive terms}$$

$$= 1 + 1000000 \times 0.01 + \text{other positive terms}$$

$$= 1 + 10000 + \text{other positive terms}$$

$$> 10000$$

Hence $(1.01)^{1000000} > 10000$

Example 4 Using binomial theorem, prove that $6^n–5n$ always leaves remainder 1 when divided by 25.

Solution For two numbers a and b if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that $6^n – 5n$ leaves remainder 1 when divided by 25, we prove that $6^n – 5n = 25k + 1$, where k is some natural number.

We have

$$(1 + a)^n = {}^nC_0 + {}^nC_1a + {}^nC_2a^2 + ... + {}^nC_na^n$$

For $a = 5$, we get

$$(1 + 5)^n = {}^nC_0 + {}^nC_15 + {}^nC_25^2 + ... + {}^nC_n5^n$$

i.e. $(6)^n = 1 + 5n + 5^2.{}^nC_2 + 5^3.{}^nC_3 + ... + 5^n$

i.e. $6^n – 5n = 1+5^2 ({}^nC_2 + {}^nC_35 + ... + 5^{n-2})$

or $6^n – 5n = 1+ 25 ({}^nC_2 + 5 .{}^nC_3 + ... + 5^{n-2})$

or $6^n – 5n = 25k+1$ where $k = {}^nC_2 + 5 .{}^nC_3 + ... + 5^{n–2}$.

This shows that when divided by 25, $6^n – 5n$ leaves remainder 1.

EXERCISE 8.1

Expand each of the expressions in Exercises 1 to 5.

1. $(1–2x)^5$ **2.** $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ **3.** $(2x – 3)^6$

4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ **5.** $\left(x+\frac{1}{x}\right)^6$

Using binomial theorem, evaluate each of the following:

6. $(96)^3$ **7.** $(102)^5$ **8.** $(101)^4$

9. $(99)^5$

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

11. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$.

12. Find $(x + 1)^6 + (x – 1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$.

13. Show that $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is a positive integer.

14. Prove that $\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$.

8.3 General and Middle Terms

1. In the binomial expansion for $(a + b)^n$, we observe that the first term is ${}^nC_0a^n$, the second term is ${}^nC_1a^{n-1}b$, the third term is ${}^nC_2a^{n-2}b^2$, and so on. Looking at the pattern of the successive terms we can say that the $(r + 1)^{th}$ term is ${}^nC_ra^{n-r}b^r$. The $(r + 1)^{th}$ term is also called the *general term* of the expansion $(a + b)^n$. It is denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_r\ a^{n-r}b^r$.

2. Regarding the middle term in the expansion $(a + b)^n$, we have

(i) If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even so $n + 1$ is odd. Therefore, the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$, i.e., $\left(\frac{n}{2}+1\right)^{th}$ term.

For example, in the expansion of $(x + 2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5^{th} term.

(ii) If n is odd, then $n +1$ is even, so there will be two middle terms in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term. So in the expansion $(2x-y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$, i.e., 4^{th} and $\left(\frac{7+1}{2}+1\right)^{th}$, i.e., 5^{th} term.

3. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{th}$, i.e., $(n+1)^{th}$ term, as $2n$ is even.

It is given by ${}^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ (constant).

This term is called the *term independent* of x or the constant term.

Example 5 Find a if the 17^{th} and 18^{th} terms of the expansion $(2+a)^{50}$ are equal.

Solution The $(r+1)^{th}$ term of the expansion $(x+y)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$.

For the 17^{th} term, we have, $r + 1 = 17$, i.e., $r = 16$

Therefore, $T_{17} = T_{16+1} = {}^{50}C_{16}\,(2)^{50-16}\, a^{16}$

$= {}^{50}C_{16}\, 2^{34}\, a^{16}.$

Similarly, $T_{18} = {}^{50}C_{17}\, 2^{33}\, a^{17}$

Given that $T_{17} = T_{18}$

So ${}^{50}C_{16}\,(2)^{34}\, a^{16} = {}^{50}C_{17}\,(2)^{33}\, a^{17}$

Therefore $$\frac{{}^{50}C_{16}\,.\,2^{34}}{{}^{50}C_{17}\,.\,2^{33}} = \frac{a^{17}}{a^{16}}$$

i.e., $$a = \frac{{}^{50}C_{16} \times 2}{{}^{50}C_{17}} = \frac{50!}{16!\,34!} \times \frac{17!\,.\,33!}{50!} \times 2 = 1$$

Example 6 Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!} 2n\, x^n$, where n is a positive integer.

Solution As $2n$ is even, the middle term of the expansion $(1 + x)^{2n}$ is $\left(\frac{2n}{2}+1\right)^{th}$, i.e., $(n + 1)^{th}$ term which is given by,

$$T_{n+1} = {}^{2n}C_n(1)^{2n-n}(x)^n = {}^{2n}C_n x^n = \frac{(2n)!}{n!\ n!}x^n$$

$$= \frac{2n(2n-1)\ (2n-2)\ ...4.3.2.1}{n!\ n!}x^n$$

$$= \frac{1.2.3.4...(2n-2)(2n-1)(2n)}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n-1)][2.4.6...(2n)]}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n-1)]2^n[1.2.3...n]}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n-1)]\,n!}{n!\ n!}2^n.x^n$$

$$= \frac{1.3.5...(2n-1)}{n!}2^n x^n$$

Example 7 Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$.

Solution Suppose x^6y^3 occurs in the $(r + 1)^{th}$ term of the expansion $(x + 2y)^9$.

Now $\quad T_{r+1} = {}^9C_r\, x^{9-r}(2y)^r = {}^9C_r\, 2^r . x^{9-r} . y^r$.

Comparing the indices of x as well as y in x^6y^3 and in T_{r+1}, we get $r = 3$.

Thus, the coefficient of x^6y^3 is

$$^9C_3\, 2^3 = \frac{9!}{3!\,6!}.2^3 = \frac{9.8.7}{3.2}.2^3 = 672.$$

Example 8 The second, third and fourth terms in the binomial expansion $(x + a)^n$ are 240, 720 and 1080, respectively. Find x, a and n.

Solution Given that second term $T_2 = 240$

We have $T_2 = {}^nC_1 x^{n-1} . a$

So ${}^nC_1 x^{n-1} . a = 240$ ... (1)

Similarly ${}^nC_2 x^{n-2} a^2 = 720$ ... (2)

and ${}^nC_3 x^{n-3} a^3 = 1080$ ... (3)

Dividing (2) by (1), we get

$$\frac{{}^nC_2 x^{n-2} a^2}{{}^nC_1 x^{n-1} a} = \frac{720}{240} \quad \text{i.e.,} \quad \frac{(n-1)!}{(n-2)!} . \frac{a}{x} = 6$$

or $$\frac{a}{x} = \frac{6}{(n-1)}$$ ... (4)

Dividing (3) by (2), we have

$$\frac{a}{x} = \frac{9}{2(n-2)}$$ ... (5)

From (4) and (5),

$$\frac{6}{n-1} = \frac{9}{2(n-2)} . \quad \text{Thus, } n = 5$$

Hence, from (1), $5x^4 a = 240$, and from (4), $\frac{a}{x} = \frac{3}{2}$

Solving these equations for a and x, we get $x = 2$ and $a = 3$.

Example 9 The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio1: 7 : 42. Find n.

Solution Suppose the three consecutive terms in the expansion of $(1 + a)^n$ are $(r – 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms.

The $(r – 1)^{th}$ term is ${}^nC_{r-2} a^{r-2}$, and its coefficient is ${}^nC_{r-2}$. Similarly, the coefficients of r^{th} and $(r + 1)^{th}$ terms are ${}^nC_{r-1}$ and ${}^nC_r$, respectively.

Since the coefficients are in the ratio 1 : 7 : 42, so we have,

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{7}, \text{ i.e., } n - 8r + 9 = 0$$ ... (1)

and $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{7}{42}, \text{ i.e., } n - 7r + 1 = 0$$ ... (2)

Solving equations(1) and (2), we get, $n = 55$.