5) Let $(1+x+2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots + a_{40}x^{40}$. Then, $a_1 + a_3 + a_5 + \ldots + a_{37}$ is equal to

(a) $2^{19}(2^{20}-21)$ (b) $2^{19}(2^{20}+21)$ [Main March 18, 2021 (I)]

(c) $2^{20}(2^{20}-21)$ (d) $2^{20}(2^{20}+21)$

Solution:

(a) $\because (1+x+2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots + a_{40}x^{40}$

Put $x=1$

$\Rightarrow 4^{20} = a_0 + a_1 + \ldots + a_{40}$ ——— (1)

Put $x=-1$.

$\Rightarrow 2^{20} = a_0 - a_1 + \ldots - a_{39} + a_{40}$ ——— (2)

Subtract (2) from (1), we get,

$$4^{20} - 2^{20} = 2(a_1 + a_3 + \ldots + a_{37} + a_{39})$$

$\Rightarrow a_1 + a_3 + \ldots + a_{37} = 2^{39} - 2^{19} - a_{39}$ ——— (3)

$\because a_{39} =$ coefficient of x^{39} in $(1+x+2x)^{20}$

$$= \frac{20!}{0!\,1!\,19!}(1)^0(1)^1(2)^{19} = 20.2^{19}$$

Substituting the value of a_{39} in equation (3); we get

$$a_1 + a_3 + \ldots + a_{37} = 2^{39} - 2^{19} - a_{39}$$

$$= 2^{39} - 2^{19} - 20.2^{19}$$

$$= 2^{39} - 2^{19}.21 = \boxed{2^{19}(2^{20}-1)}$$