Quadratic Equation

$ax^2+bx+c=0$ $\quad a,b,c \in \mathbb{C}$
$a \neq 0.$

If $a \in \mathbb{R}$, Without loss of generality $a>0$, because if a is negative, we can simply multiply the equation by (-1)

This quadratic equation is having exactly 2 solutions:

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Derivation:

Let $\frac{-b-\sqrt{b^2-4ac}}{2a} = \alpha$, $\frac{-b+\sqrt{b^2-4ac}}{2a} = \beta$

$$ax^2+bx+c=0$$

$$\Rightarrow a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0$$

$$\Rightarrow a\left(x^2+2\cdot x\cdot\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)=0$$

$$\Rightarrow a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right\}=0$$ — proved.

$D := b^2-4ac$

$\hookrightarrow$ Discriminant of the polynomial ax^2+bx+c

$$\alpha = \frac{-b-\sqrt{D}}{2a}, \quad \beta = \frac{-b+\sqrt{D}}{2a}$$

If $a,b,c \in \mathbb{Q}$,

1) D is a square, $\Rightarrow \alpha,\beta \in \mathbb{Q}$
2) D is not a square $\Rightarrow \alpha,\beta$ are conjugate surds

$a,b,c \in \mathbb{R}$, $a>0$

1) If $D>0$, then α, β are distinct and real
2) If $D=0$, then $\alpha=\beta$, are real
3) If $D<0$, then α,β are distinct and complex conjugates of each other

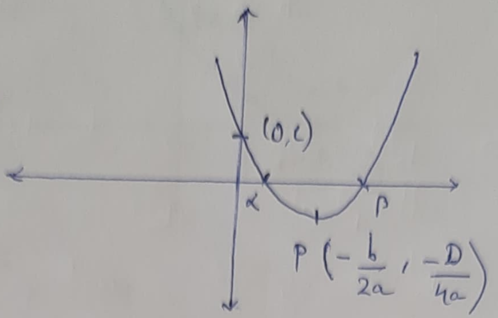
Let's draw graphs:

$y = ax^2 + bx + c$ $\quad a, b, c \in \mathbb{R}$

$a > 0$

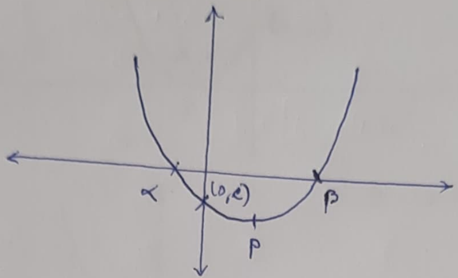
<u>Case – I:</u> $D > 0$

i) $b < 0,\ c > 0$



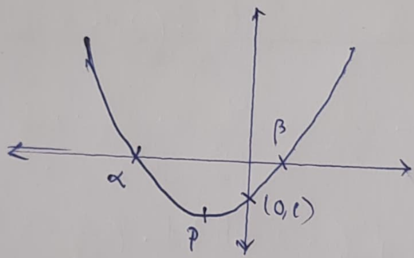
α, β both are real and distinct.

ii) $b < 0,\ c < 0$



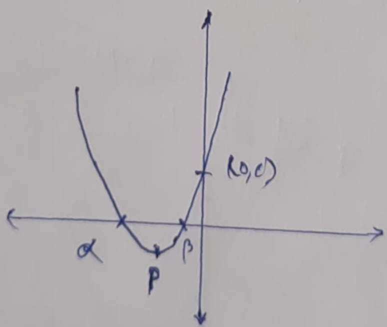
α, β both are real and distinct

iii) $c < 0,\ b > 0$



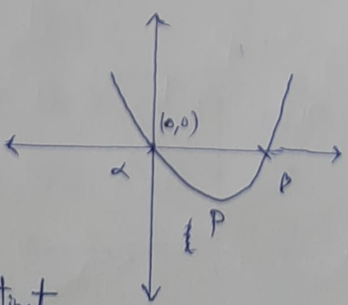
α, β both are real and distinct.

iv) $c > 0,\ b > 0$.



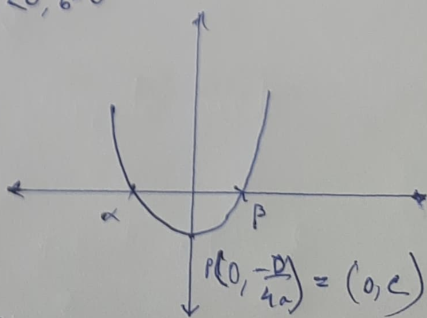
α, β both are real and distinct.

v) $c = 0,\ b > 0$



α, β are real and distinct

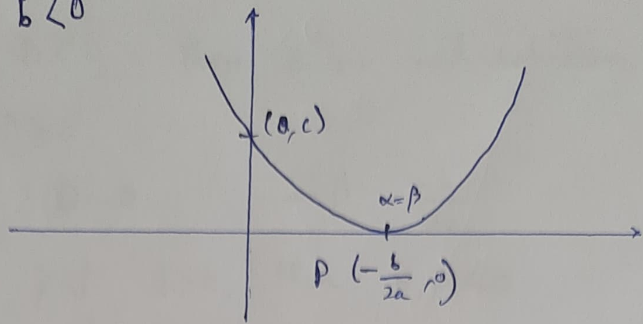
vi) $c < 0,\ b = 0$



α, β are real and distinct.

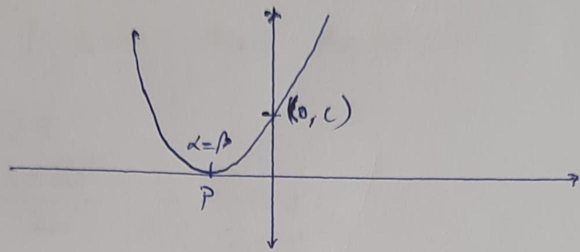
Case-II : $D = 0$

i) $c > 0,\ b < 0$



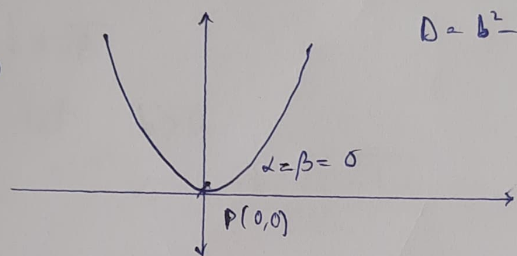
$\alpha = \beta = -\frac{b}{2a}$

ii) $c > 0,\ b > 0$



$\alpha = \beta$

iii) $c = 0,\ b = 0$



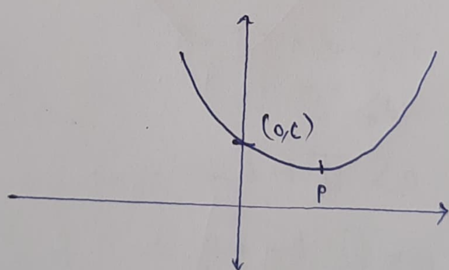
$D = b^2 - 4ac = 0 \Rightarrow b^2 = 4ac \quad ac \geq 0$

$c \geq 0$

$\alpha = \beta = 0$

Case-III : $D < 0$

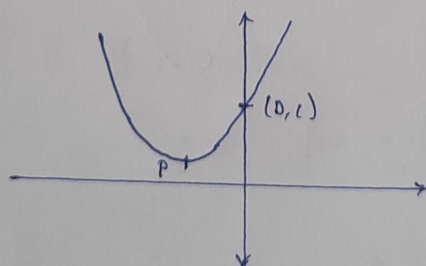
i) $b < 0,\ c > 0$



No real solution

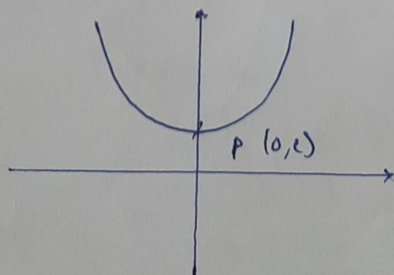
α, β are not real numbers

ii) $b > 0,\ c > 0$



No real solution

iii) $b = 0,\ c > 0$



No real solution

Sign of α, β

$a, b, c \in \mathbb{R}$

For $D < 0$, there is no real solution

Consider $D \geq 0$.

Case - I: $D = 0$ $\qquad \alpha = \beta = \frac{-b}{2a}$

i) if $b > 0$, then $\alpha, \beta < 0$

ii) if $b < 0$, then $\alpha, \beta > 0$

iii) if $b = 0$ then $\alpha = \beta = 0$

Case - II: $D > 0$

$\frac{-b \pm \sqrt{D}}{2a}$ $\qquad$ $a > 0$, $2a > 0$

$-b \pm \sqrt{D}$

i) Let $b > 0$

$\alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0$

if $c > 0$, then

$D = b^2 - 4ac < b^2$

$\therefore \sqrt{D} < b$ as $D, b > 0$

$\therefore -b + \sqrt{D} < 0$

$\therefore \beta = \frac{-b + \sqrt{D}}{2a} < 0$

if $c = 0$, then

$\beta = 0 \qquad \beta = \frac{-b + \sqrt{b^2}}{2a}$

$= \frac{-b + b}{2a}$

$= 0$

if $c < 0$, then

$D = b^2 - 4ac > b^2$

$\therefore \sqrt{D} > b$

$\therefore \beta = \frac{-b + \sqrt{D}}{2a} > 0$

ii) if $b<0$

$$\beta = \frac{-b+\sqrt{D}}{2a} > 0$$

if $c>0$ $\quad D = b^2-4ac < b^2$

$\therefore \sqrt{D} < -b$

as $D>0$

and $b<0$

$-b-\sqrt{D} > 0$

$$\therefore \alpha = \frac{-b-\sqrt{D}}{2a} > 0$$

if $c=0$ then $\alpha = 0$

$$\frac{-b-\sqrt{b^2}}{2a} = \frac{-b+b}{2a} = 0$$

if $c<0$, $D>b^2$

$\therefore \sqrt{D} > -b$

$\therefore \alpha < 0$

iii) if $b=0$,

$$\frac{\pm\sqrt{D}}{2a}$$

$D>0$

$$\alpha = -\frac{\sqrt{D}}{2a} < 0$$

and $$\beta = \frac{\sqrt{D}}{2a} > 0$$