complicated trajectories. Yet, if the total external force acting on the system is zero, the centre of mass moves with *a* constant velocity, i.e., moves uniformly in a straight line like a free particle.

The vector Eq. (7.18a) is equivalent to three scalar equations,

$P_x = c_1$, $P_y = c_2$ and $P_z = c_3$ (7.18 b)

Here P_x, P_y and P_z are the components of the total linear momentum vector **P** along the *x*–, *y*– and *z*–axes respectively; c_1, c_2 and c_3 are constants.

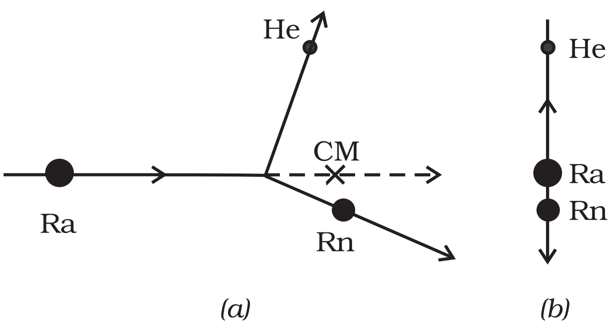


Fig. 7.13 (a) *A heavy nucleus radium (Ra) splits into a lighter nucleus radon (Rn) and an alpha particle (nucleus of helium atom). The CM of the system is in uniform motion.*
(b) *The same spliting of the heavy nucleus radium (Ra) with the centre of mass at rest. The two product particles fly back to back.*

As an example, let us consider the radioactive decay of a moving unstable particle, like the nucleus of radium. A radium nucleus disintegrates into a nucleus of radon and an alpha particle. The forces leading to the decay are internal to the system and the external forces on the system are negligible. So the total linear momentum of the system is the same before and after decay. The two particles produced in the decay, the radon nucleus and the alpha particle, move in different directions in such a way that their centre of mass moves along the same path along which the original decaying radium nucleus was moving [Fig. 7.13(a)].

If we observe the decay from the frame of reference in which the centre of mass is at rest, the motion of the particles involved in the decay looks particularly simple; the product particles

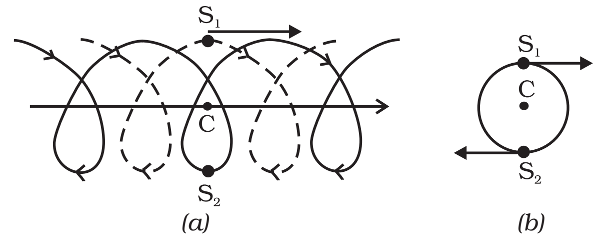


Fig. 7.14 (a) *Trajectories of two stars, S_1 (dotted line) and S_2 (solid line) forming a binary system with their centre of mass C in uniform motion.*
(b) *The same binary system, with the centre of mass C at rest.*

move back to back with their centre of mass remaining at rest as shown in Fig.7.13 (b).

In many problems on the system of particles, as in the above radioactive decay problem, it is convenient to work in the centre of mass frame rather than in the laboratory frame of reference.

In astronomy, binary (double) stars is a common occurrence. If there are no external forces, the centre of mass of a double star moves like a free particle, as shown in Fig.7.14 (a). The trajectories of the two stars of equal mass are also shown in the figure; they look complicated. If we go to the centre of mass frame, then we find that there the two stars are moving in a circle, about the centre of mass, which is at rest. Note that the position of the stars have to be diametrically opposite to each other [Fig. 7.14(b)]. Thus in our frame of reference, the trajectories of the stars are a combination of (i) uniform motion in a straight line of the centre of mass and (ii) circular orbits of the stars about the centre of mass.

As can be seen from the two examples, **separating** the motion of different parts of a system into motion **of the centre of mass and motion about the centre of mass** is a very useful technique that helps in understanding the motion of the system.

7.5 VECTOR PRODUCT OF TWO VECTORS

We are already familiar with vectors and their use in physics. In chapter 6 (Work, Energy, Power) we defined the scalar product of two vectors. An important physical quantity, work, is defined as a scalar product of two vector quantities, force and displacement.

We shall now define another product of two vectors. This product is a vector. Two important quantities in the study of rotational motion, namely, moment of a force and angular momentum, are defined as vector products.

Definition of Vector Product

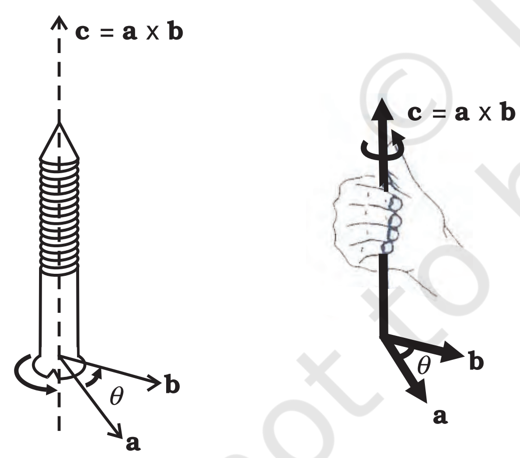
A vector product of two vectors **a** and **b** is a vector **c** such that

(i) magnitude of **c** = $c = ab \sin\theta$ where a and b are magnitudes of **a** and **b** and θ is the angle between the two vectors.

(ii) **c** is perpendicular to the plane containing **a** and **b**.

(iii) if we take a right handed screw with its head lying in the plane of **a** and **b** and the screw perpendicular to this plane, and if we turn the head in the direction from **a** to **b**, then the tip of the screw advances in the direction of **c**. This right handed screw rule is illustrated in Fig. 7.15a.

Alternately, if one curls up the fingers of right hand around a line perpendicular to the plane of the vectors **a** and **b** and if the fingers are curled up in the direction from **a** to **b**, then the stretched thumb points in the direction of **c**, as shown in Fig. 7.15b.



(a) (b)

Fig. 7.15 (a) Rule of the right handed screw for defining the direction of the vector product of two vectors.
(b) Rule of the right hand for defining the direction of the vector product.

A simpler version of the right hand rule is the following : Open up your right hand palm and curl the fingers pointing from **a** to **b**. Your stretched thumb points in the direction of **c**.

It should be remembered that there are two angles between any two vectors **a** and **b** . In Fig. 7.15 (a) or (b) they correspond to θ (as shown) and ($360^0 - \theta$). While applying either of the above rules, the rotation should be taken through the smaller angle ($<180^0$) between **a** and **b**. It is θ here.

Because of the cross (×) used to denote the vector product, it is also referred to as cross product.

- Note that scalar product of two vectors is commutative as said earlier, **a**.**b** = **b**.**a**

The vector product, however, is not commutative, i.e. $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

The magnitude of both **a**×**b** and **b**×**a** is the same ($ab \sin\theta$); also, both of them are perpendicular to the plane of **a** and **b**. But the rotation of the right-handed screw in case of **a**×**b** is from **a** to **b**, whereas in case of **b**×**a** it is from **b** to **a**. This means the two vectors are in opposite directions. We have

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

- Another interesting property of a vector product is its behaviour under reflection. Under reflection (i.e. on taking the plane mirror image) we have $x \to -x, y \to -y$ and $z \to -z$. As a result all the components of a vector change sign and thus $a \to -a$, $b \to -b$. What happens to $\mathbf{a} \times \mathbf{b}$ under reflection?

$$\mathbf{a} \times \mathbf{b} \to (-\mathbf{a}) \times (-\mathbf{b}) = \mathbf{a} \times \mathbf{b}$$

Thus, $\mathbf{a} \times \mathbf{b}$ does not change sign under reflection.

- Both scalar and vector products are distributive with respect to vector addition. Thus,

$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

- We may write **c** = **a**×**b** in the component form. For this we first need to obtain some elementary cross products:

(i) **a**×**a** = **0** (**0** is a null vector, i.e. a vector with zero magnitude)

This follows since magnitude of $\mathbf{a} \times \mathbf{a}$ is $a^2 \sin 0° = 0$.

From this follow the results

(i) $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}, \ \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \mathbf{0}, \ \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$

(ii) $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$

Note that the magnitude of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is $\sin 90^0$ or 1, since $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ both have unit magnitude and the angle between them is 90^0. Thus, $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is a unit vector. A unit vector perpendicular to the plane of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ and related to them by the right hand screw rule is $\hat{\mathbf{k}}$. Hence, the above result. You may verify similarly,

$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$

From the rule for commutation of the cross product, it follows:

$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \ \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \ \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$

Note if $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ occur cyclically in the above vector product relation, the vector product is positive. If $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ do not occur in cyclic order, the vector product is negative.

Now,

$$\mathbf{a} \times \mathbf{b} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}) \times (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}})$$

$$= a_x b_y \hat{\mathbf{k}} - a_x b_z \hat{\mathbf{j}} - a_y b_x \hat{\mathbf{k}} + a_y b_z \hat{\mathbf{i}} + a_z b_x \hat{\mathbf{j}} - a_z b_y \hat{\mathbf{i}}$$

$$= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

We have used the elementary cross products in obtaining the above relation. The expression for $\mathbf{a} \times \mathbf{b}$ can be put in a determinant form which is easy to remember.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

▶ ***Example 7.4*** *Find the scalar and vector products of two vectors.* $\mathbf{a} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ *and* $\mathbf{b} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$

Answer

$$\mathbf{a} \cdot \mathbf{b} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \cdot (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$
$$= -6 - 4 - 15$$
$$= -25$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix} = 7\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

Note $\mathbf{b} \times \mathbf{a} = -7\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ ◀

7.6 ANGULAR VELOCITY AND ITS RELATION WITH LINEAR VELOCITY

In this section we shall study what is angular velocity and its role in rotational motion. We have seen that every particle of a rotating body moves in a circle. The linear velocity of the particle is related to the angular velocity. The relation between these two quantities involves a vector product which we learnt about in the last section.

Let us go back to Fig. 7.4. As said above, in rotational motion of a rigid body about a fixed axis, every particle of the body moves in a circle,

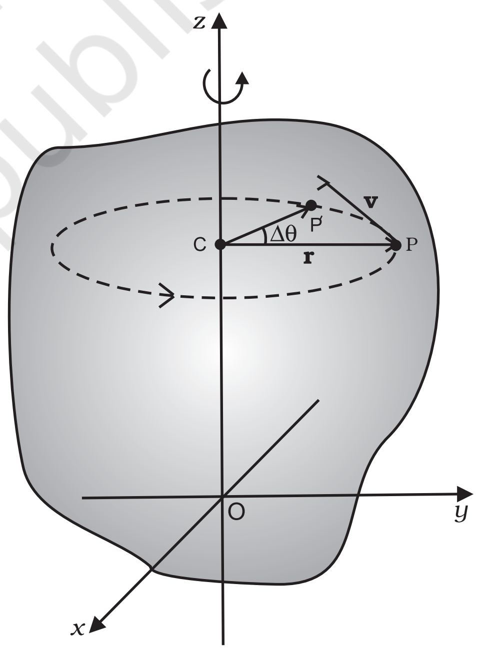


Fig. 7.16 *Rotation about a fixed axis. (A particle (P) of the rigid body rotating about the fixed (z-) axis moves in a circle with centre (C) on the axis.)*

which lies in a plane perpendicular to the axis and has its centre on the axis. In Fig. 7.16 we redraw Fig. 7.4, showing a typical particle (at a point P) of the rigid body rotating about a fixed axis (taken as the *z*-axis). The particle describes a circle with a centre C on the axis. The radius of the circle is r, the perpendicular distance of the point P from the axis. We also show the linear velocity vector **v** of the particle at P. It is along the tangent at P to the circle.

Let P′ be the position of the particle after an interval of time Δt (Fig. 7.16). The angle PCP′ describes the angular displacement $\Delta\theta$ of the particle in time Δt. The average angular velocity of the particle over the interval Δt is $\Delta\theta/\Delta t$. As Δt tends to zero (i.e. takes smaller and smaller values), the ratio $\Delta\theta/\Delta t$ approaches a limit which is the instantaneous angular velocity $d\theta/dt$ of the particle at the position P. We denote the **instantaneous angular velocity** by ω (the Greek letter omega). We know from our study of circular motion that the magnitude of linear velocity υ of a particle moving in a circle is related to the angular velocity of the particle ω by the simple relation $\upsilon = \omega r$, where r is the radius of the circle.

We observe that at any given instant the relation $\upsilon = \omega r$ applies to all particles of the rigid body. Thus for a particle at a perpendicular distance r_i from the fixed axis, the linear velocity at a given instant υ_i is given by

$$\upsilon_i = \omega r_i \tag{7.19}$$

The index i runs from 1 to n, where n is the total number of particles of the body.

For particles on the axis, $r = 0$, and hence $\upsilon = \omega r = 0$. Thus, particles on the axis are stationary. This verifies that the axis is *fixed*.

Note that we use the same angular velocity ω for all the particles. **We therefore, refer to ω as the angular velocity of the whole body**.

We have characterised pure translation of a body by all parts of the body having the same velocity at any instant of time. Similarly, we may characterise pure rotation by all parts of the body having the same angular velocity at any instant of time. Note that this characterisation of the rotation of a rigid body about a fixed axis is **just another way** of saying as in Sec. 7.1 that each particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has the centre on the axis.

In our discussion so far the angular velocity appears to be a scalar. In fact, it is a vector. We shall not justify this fact, but we shall accept it. For rotation about a fixed axis, the angular velocity vector lies along the axis of rotation, and points out in the direction in which a right handed screw would advance, if the head of the screw is rotated with the body. (See Fig. 7.17a).

The magnitude of this vector is $\omega = d\theta/dt$ referred as above.

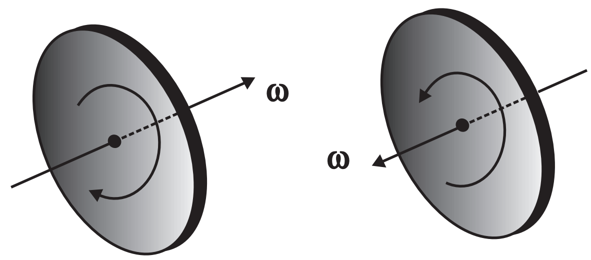


Fig. 7.17 (a) *If the head of a right handed screw rotates with the body, the screw advances in the direction of the angular velocity* **ω**. *If the sense (clockwise or anticlockwise) of rotation of the body changes, so does the direction of* **ω**.

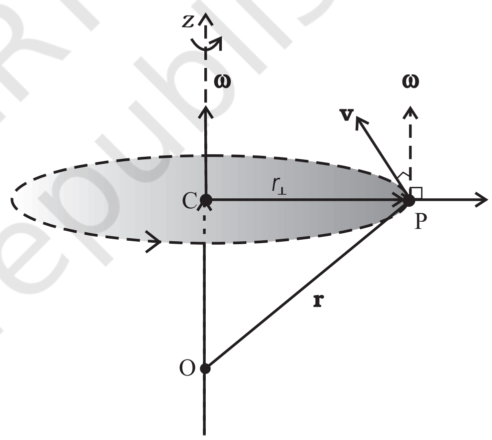


Fig. 7.17 (b) *The angular velocity vector* **ω** *is directed along the fixed axis as shown. The linear velocity of the particle at P is* **v** = **ω** × **r**. *It is perpendicular to both* **ω** *and* **r** *and is directed along the tangent to the circle described by the particle.*

We shall now look at what the vector product **ω** × **r** corresponds to. Refer to Fig. 7.17(b) which is a part of Fig. 7.16 reproduced to show the path of the particle P. The figure shows the vector **ω** directed along the fixed (z–) axis and also the position vector **r** = **OP** of the particle at P of the rigid body with respect to the origin O. Note that the origin is chosen to be on the axis of rotation.

Now $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{OP} = \boldsymbol{\omega} \times (\mathbf{OC} + \mathbf{CP})$

But $\boldsymbol{\omega} \times \mathbf{OC} = 0$ as $\boldsymbol{\omega}$ is along **OC**

Hence $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{CP}$

The vector $\boldsymbol{\omega} \times \mathbf{CP}$ is perpendicular to $\boldsymbol{\omega}$, i.e. to the z-axis and also to **CP**, the radius of the circle described by the particle at P. It is therefore, along the tangent to the circle at P. Also, the magnitude of $\boldsymbol{\omega} \times \mathbf{CP}$ is ω (CP) since $\boldsymbol{\omega}$ and **CP** are perpendicular to each other. We shall denote **CP** by $\mathbf{r}_\perp$ and not by **r**, as we did earlier.

Thus, $\boldsymbol{\omega} \times \mathbf{r}$ is a vector of magnitude $\omega r_\perp$ and is along the tangent to the circle described by the particle at P. The linear velocity vector **v** at P has the same magnitude and direction. Thus,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \qquad (7.20)$$

In fact, the relation, Eq. (7.20), holds good even for rotation of a rigid body with one point fixed, such as the rotation of the top [Fig. 7.6(a)]. In this case **r** represents the position vector of the particle with respect to the fixed point taken as the origin.

We note that **for rotation about a fixed axis, the direction of the vector ω does not change with time. Its magnitude may, however, change from instant to instant. For the more general rotation, both the magnitude and the direction of ω may change from instant to instant.**

7.6.1 Angular acceleration

You may have noticed that we are developing the study of rotational motion along the lines of the study of translational motion with which we are already familiar. Analogous to the kinetic variables of linear displacement (**s**) and velocity (**v**) in translational motion, we have angular displacement (**θ**) and angular velocity (**ω**) in rotational motion. It is then natural to define in rotational motion the concept of angular acceleration in analogy with linear acceleration defined as the time rate of change of velocity in translational motion. We define angular acceleration **α** as the time rate of change of angular velocity; Thus,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} \qquad (7.21)$$

If the axis of rotation is fixed, the direction of **ω** and hence, that of **α** is fixed. In this case the vector equation reduces to a scalar equation

$$\alpha = \frac{d\omega}{dt} \qquad (7.22)$$

7.7 TORQUE AND ANGULAR MOMENTUM

In this section, we shall acquaint ourselves with two physical quantities (torque and angular momentum) which are defined as vector products of two vectors. These as we shall see, are especially important in the discussion of motion of systems of particles, particularly rigid bodies.

7.7.1 Moment of force (Torque)

We have learnt that the motion of a rigid body, in general, is a combination of rotation and translation. If the body is fixed at a point or along a line, it has only rotational motion. We know that force is needed to change the translational state of a body, i.e. to produce linear acceleration. We may then ask, what is the analogue of force in the case of rotational motion? To look into the question in a concrete situation let us take the example of opening or closing of a door. A door is a rigid body which can rotate about a fixed vertical axis passing through the hinges. What makes the door rotate? It is clear that unless a force is applied the door does not rotate. But any force does not do the job. A force applied to the hinge line cannot produce any rotation at all, whereas a force of given magnitude applied at right angles to the door at its outer edge is most effective in producing rotation. It is not the force alone, but how and where the force is applied is important in rotational motion.

The rotational analogue of force in linear motion is **moment of force**. It is also referred to as **torque** or **couple**. (We shall use the words moment of force and torque interchangeably.) We shall first define the moment of force for the special case of a single particle. Later on we shall extend the concept to systems of particles including rigid bodies. We shall also relate it to a change in the state of rotational motion, i.e. is angular acceleration of a rigid body.