Chain Rule: It is also called

- **Composite Function Rule or**
- **Function of a Function Rule**

Theorem:

Let $y = f(u)$ be a function of u and in turn let $u = g(x)$ be a function of x so that $y = f\left(g(x)\right) = (fog)(x)$.

Then $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$.

Proof:

In the above function $u = g(x)$ is known as the inner function and f is known as the outer function. Note that, ultimately, y is a function of x.

Now $\Delta u = g(x+\Delta x) - g(x)$

Therefore, $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x} = \frac{f(u+\Delta u) - f(u)}{\Delta u} \times \frac{g(x+\Delta x) - g(x)}{\Delta x}$.

Note that $\Delta u \to 0$ as $\Delta x \to 0$

$$
\begin{aligned}
\text{Therefore, } \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta u} \times \frac{\Delta y}{\Delta x}\right) \\
&= \lim_{\Delta u \to 0}\left(\frac{\Delta y}{\Delta u}\right) . \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) \\
&= \lim_{\Delta u \to 0} \frac{f(u+\Delta u) - f(u)}{\Delta u} \times \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(u) \times u'(x) \\
&= f'\left(g(x)\right) g'(x) \quad \text{or} \quad \boxed{\frac{d}{dx}\left(f\left(g(x)\right) = f'\left(g(x)\right) g'(x)\right.}.
\end{aligned}
$$

Remark:

Thus, to differentiate a function of a function $y = f\left(g(x)\right)$, we have to take the derivative of the outer function f regarding the argument $g(x) = u$, and multiply the derivative of the inner function $g(x)$ with respect to the independent variable x. The variable u is known as **intermediate argument**.

Examples:

Example-1:- Find the derivative of tan $(2x + 3)$.

Solution Let $f(x) = \tan(2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \circ u)(x) = v(u(x)) = v(2x + 3) = \tan(2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2(2x+3)$$

Example-2: Find the derivative of tan $(2x + 3)$.

Sol: Let $f(x) = \tan(2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \circ u)(x) = v(u(x)) = v(2x + 3) = \tan(2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2(2x+3)$$

Derivaitve of Inverse Function:

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'(f^{-1}(x)) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x)) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}. \qquad \text{eq-1}$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}. \qquad \text{eq-2}$$

Important Formulas:

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2 - 1}}$$

Examples from NCERT done by prof in class:

Example 26 Find the derivative of f given by $f(x) = \sin^{-1} x$ assuming it exists.

Solution Let $y = \sin^{-1} x$. Then, $x = \sin y$.

Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Observe that this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$, i.e., $x \in (-1, 1)$.

To make this result a bit more attractive, we carry out the following manipulation. Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1} x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y$ is positive and hence $\cos y = \sqrt{1-x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Example 27 Find the derivative of f given by $f(x) = \tan^{-1} x$ assuming it exists.

Solution Let $y = \tan^{-1} x$. Then, $x = \tan y$.

Differentiating both sides w.r.t. x, we get

$$1 = \sec^2 y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+(\tan(\tan^{-1} x))^2} = \frac{1}{1+x^2}$$