Question 5. The sides of a triangle are $x^2 + x + 1, 2x + 1$ and $x^2 - 1$, prove that the greatest angle is 120°

Solution.
Let $a = x^2 + x + 1, b = 2x + 1$ and $c = x^2 - 1$.

First of all, we have to decide which side is the greatest.
We know that in a triangle, the length of each side is greater than zero.
Therefore, we have $b = 2x + 1 > 0$ and $c = x^2 - 1 > 0$.
$\Rightarrow \quad x > -1/2$ and $x^2 > 1 \Rightarrow x > -1/2$ and $x < -1$ or $x > 1 \Rightarrow x > 1$
Now, $a = x^2 + x + 1 = (x + 1/2)^2 + (3/4)$ is always positive.
Thus, all sides a, b and c are positive when $x > 1$.
Now, $x > 1 \Rightarrow x^2 > x$

$$\Rightarrow x^2 + x + 1 > x + x + 1$$
$$\Rightarrow x^2 + x + 1 > 2x + 1 \Rightarrow a > b$$

Also, when $x > 1$,

$$x^2 + x + 1 > x^2 - 1 \Rightarrow a > c$$

Thus, $a = x^2 + x + 1$ is the greatest side and the angle A opposite to this side is the greatest angle.

$$\begin{aligned}
\therefore \quad \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{(2x+1)^2 + \left(x^2 - 1\right)^2 - \left(x^2 + x + 1\right)^2}{2(2x+1)\left(x^2 - 1\right)} \\
&= \frac{-2x^3 - x^2 + 2x + 1}{2\left(2x^3 + x^2 - 2x - 1\right)} = -\frac{1}{2} \\
&= \cos 120^\circ \\
\Rightarrow \quad A &= 120^\circ
\end{aligned}$$