Table 2.4 Range and order of masses

Object	Mass (kg)
Electron	10^{-30}
Proton	10^{-27}
Uranium atom	10^{-25}
Red blood cell	10^{-13}
Dust particle	10^{-9}
Rain drop	10^{-6}
Mosquito	10^{-5}
Grape	10^{-3}
Human	10^{2}
Automobile	10^{3}
Boeing 747 aircraft	10^{8}
Moon	10^{23}
Earth	10^{25}
Sun	10^{30}
Milky way galaxy	10^{41}
Observable Universe	10^{55}

2.5 MEASUREMENT OF TIME

To measure any time interval we need a clock. We now use an **atomic standard of time**, which is based on the periodic vibrations produced in a cesium atom. This is the basis of the **caesium clock**, sometimes called **atomic clock**, used in the national standards. Such standards are available in many laboratories. In the caesium atomic clock, the second is taken as the time needed for 9,192,631,770 vibrations of the radiation corresponding to the transition between the two hyperfine levels of the ground state of caesium-133 atom. The vibrations of the caesium atom regulate the rate of this caesium atomic clock just as the vibrations of a balance wheel regulate an ordinary wristwatch or the vibrations of a small quartz crystal regulate a quartz wristwatch.

The caesium atomic clocks are very accurate. In principle they provide portable standard. The national standard of time interval 'second' as well as the frequency is maintained through four cesium atomic clocks. A caesium atomic clock is used at the National Physical Laboratory (NPL), New Delhi to maintain the Indian standard of time.

In our country, the NPL has the responsibility of maintenance and improvement of physical standards, including that of time, frequency, etc. Note that the Indian Standard Time (IST) is linked to this set of atomic clocks. The efficient caesium atomic clocks are so accurate that they impart the uncertainty in time realisation as $\pm 1 \times 10^{-15}$, i.e. 1 part in 10^{15}. This implies that the uncertainty gained over time by such a device is less than 1 part in 10^{15}; they lose or gain no more than 32 μs in one year. In view of the tremendous accuracy in time measurement, the SI unit of length has been expressed in terms the path length light travels in certain interval of time (1/299, 792, 458 of a second) (Table 2.1).

The time interval of events that we come across in the universe vary over a very wide range. Table 2.5 gives the range and order of some typical time intervals.

You may notice that there is an interesting coincidence between the numbers appearing in Tables 2.3 and 2.5. Note that the ratio of the longest and shortest lengths of objects in our universe is about 10^{41}. Interestingly enough, the ratio of the longest and shortest time intervals associated with the events and objects in our universe is also about 10^{41}. This number, 10^{41} comes up again in Table 2.4, which lists typical masses of objects. The ratio of the largest and smallest masses of the objects in our universe is about $(10^{41})^2$. Is this a curious coincidence between these large numbers purely accidental ?

2.6 ACCURACY, PRECISION OF INSTRUMENTS AND ERRORS IN MEASUREMENT

Measurement is the foundation of all experimental science and technology. The result of every measurement by any measuring instrument contains some uncertainty. This uncertainty is called **error**. Every calculated quantity which is based on measured values, also has an error. We shall distinguish between two terms: **accuracy** and **precision**. The accuracy of a measurement is a measure of how close the measured value is to the true value of the quantity. Precision tells us to what resolution or limit the quantity is measured.

The accuracy in measurement may depend on several factors, including the limit or the resolution of the measuring instrument. For example, suppose the true value of a certain length is near 3.678 cm. In one experiment, using a measuring instrument of resolution 0.1 cm, the measured value is found to be 3.5 cm, while in another experiment using a measuring device of greater resolution, say 0.01 cm, the length is determined to be 3.38 cm. The first measurement has more accuracy (because it is

Table 2.5 **Range and order of time intervals**

Event	Time interval(s)
Life-span of most unstable particle	10^{-24}
Time required for light to cross a nuclear distance	10^{-22}
Period of x-rays	10^{-19}
Period of atomic vibrations	10^{-15}
Period of light wave	10^{-15}
Life time of an excited state of an atom	10^{-8}
Period of radio wave	10^{-6}
Period of a sound wave	10^{-3}
Wink of eye	10^{-1}
Time between successive human heart beats	10^{0}
Travel time for light from moon to the Earth	10^{0}
Travel time for light from the Sun to the Earth	10^{2}
Time period of a satellite	10^{4}
Rotation period of the Earth	10^{5}
Rotation and revolution periods of the moon	10^{6}
Revolution period of the Earth	10^{7}
Travel time for light from nearest star	10^{8}
Average human life-span	10^{9}
Age of Egyptian pyramids	10^{11}
Time since dinosaurs became extinct	10^{15}
Age of the universe	10^{17}

closer to the true value) but less precision (its resolution is only 0.1 cm), while the second measurement is less accurate but more precise. Thus every measurement is approximate due to errors in measurement. In general, the errors in measurement can be broadly classified as (a) systematic errors and (b) random errors.

Systematic errors

The **systematic errors** are those errors that tend to be in one direction, either positive or negative. Some of the sources of systematic errors are :

(a) **Instrumental errors** that arise from the errors due to imperfect design or calibration of the measuring instrument, zero error in the instrument, etc. For example, the temperature graduations of a thermometer may be inadequately calibrated (it may read 104 °C at the boiling point of water at STP whereas it should read 100 °C); in a vernier callipers the zero mark of vernier scale may not coincide with the zero mark of the main scale, or simply an ordinary metre scale may be worn off at one end.

(b) **Imperfection in experimental technique or procedure** To determine the temperature of a human body, a thermometer placed under the armpit will always give a temperature lower than the actual value of the body temperature. Other external conditions (such as changes in temperature, humidity, wind velocity, etc.) during the experiment may systematically affect the measurement.

(c) **Personal errors** that arise due to an individual's bias, lack of proper setting of the apparatus or individual's carelessness in taking observations without observing proper precautions, etc. For example, if you, by habit, always hold your head a bit too far to the right while reading the position of a needle on the scale, you will introduce an error due to **parallax**.

Systematic errors can be minimised by improving experimental techniques, selecting better instruments and removing personal bias as far as possible. For a given set-up, these errors may be estimated to a certain extent and the necessary corrections may be applied to the readings.

Random errors

The **random errors** are those errors, which occur irregularly and hence are random with respect to sign and size. These can arise due to random and unpredictable fluctuations in experimental conditions (e.g. unpredictable fluctuations in temperature, voltage supply, mechanical vibrations of experimental set-ups, etc), personal (unbiased) errors by the observer taking readings, etc. For example, when the same person repeats the same observation, it is very likely that he may get different readings everytime.

Least count error

The smallest value that can be measured by the measuring instrument is called its **least count**. All the readings or measured values are good only up to this value.

The **least count error** is the error associated with the resolution of the instrument. For example, a vernier callipers has the least count as 0.01 cm; a spherometer may have a least count of 0.001 cm. Least count error belongs to the category of random errors but within a limited size; it occurs with both systematic and random errors. If we use a metre scale for measurement of length, it may have graduations at 1 mm division scale spacing or interval.

Using instruments of higher precision, improving experimental techniques, etc., we can reduce the least count error. Repeating the observations several times and taking the arithmetic mean of all the observations, the mean value would be very close to the true value of the measured quantity.

2.6.1 Absolute Error, Relative Error and Percentage Error

(a) Suppose the values obtained in several measurements are $a_1, a_2, a_3...., a_n$. The arithmetic mean of these values is taken as the best possible value of the quantity under the given conditions of measurement as :

$$a_{mean} = (a_1+a_2+a_3+...+a_n) / n \quad (2.4)$$

or,

$$a_{mean} = \sum_{i=1}^{n} a_i / n \quad (2.5)$$

This is because, as explained earlier, it is reasonable to suppose that individual measurements are as likely to overestimate as to underestimate the true value of the quantity.

The magnitude of the difference between the individual measurement and the true value of the quantity is called the absolute error of the measurement. This is denoted by $|\Delta a|$. In absence of any other method of knowing true value, we considered arithmatic mean as the true value. Then the errors in the individual measurement values from the true value, are

$\Delta a_1 = a_1 - a_{mean}$,
$\Delta a_2 = a_2 - a_{mean}$,
....
....
$\Delta a_n = a_n - a_{mean}$

The Δa calculated above may be positive in certain cases and negative in some other cases. But absolute error $|\Delta a|$ will always be positive.

(b) The arithmetic mean of all the *absolute errors* is taken as the *final* or *mean absolute error* of the value of the physical quantity a. It is represented by Δa_{mean}.

Thus,

$$\Delta a_{mean} = (|\Delta a_1|+|\Delta a_2|+|\Delta a_3|+...+|\Delta a_n|)/n \quad (2.6)$$

$$= \sum_{i=1}^{n} |\Delta a_i|/n \quad (2.7)$$

If we do a single measurement, the value we get may be in the range $a_{mean} \pm \Delta a_{mean}$

i.e. $a = a_{mean} \pm \Delta a_{mean}$

or,

$$a_{mean} - \Delta a_{mean} \le a \le a_{mean} + \Delta a_{mean} \quad (2.8)$$

This implies that any measurement of the physical quantity a is likely to lie between $(a_{mean} + \Delta a_{mean})$ and $(a_{mean} - \Delta a_{mean})$.

(c) Instead of the absolute error, we often use the **relative error** or the **percentage error** (δa). ***The relative error is the ratio of the mean absolute error Δa_{mean} to the mean value a_{mean} of the quantity measured.***

Relative error = $\Delta a_{mean}/a_{mean}$ (2.9)

When the relative error is expressed in per cent, it is called the **percentage error** (δa).

Thus, Percentage error

$$\delta a = (\Delta a_{mean}/a_{mean})\times 100\% \quad (2.10)$$

Let us now consider an example.

Example 2.6 Two clocks are being tested against a standard clock located in a national laboratory. At 12:00:00 noon by the standard clock, the readings of the two clocks are :

	Clock 1	Clock 2
Monday	12:00:05	10:15:06
Tuesday	12:01:15	10:14:59
Wednesday	11:59:08	10:15:18
Thursday	12:01:50	10:15:07
Friday	11:59:15	10:14:53
Saturday	12:01:30	10:15:24
Sunday	12:01:19	10:15:11

If you are doing an experiment that requires precision time interval measurements, which of the two clocks will you prefer ?

Answer The range of variation over the seven days of observations is 162 s for clock 1, and 31 s for clock 2. The average reading of clock 1 is much closer to the standard time than the average reading of clock 2. The important point is that a clock's zero error is not as significant for precision work as its variation, because a 'zero-error' can always be easily corrected. Hence clock 2 is to be preferred to clock 1. ◀

Example 2.7 We measure the period of oscillation of a simple pendulum. In successive measurements, the readings turn out to be 2.63 s, 2.56 s, 2.42 s, 2.71s and 2.80 s. Calculate the absolute errors, relative error or percentage error.

Answer The mean period of oscillation of the pendulum

$$T = \frac{(2.63+2.56+2.42+2.71+2.80)\text{s}}{5}$$

$$= \frac{13.12}{5}\ \text{s}$$

$$= 2.624\ \text{s}$$

$$= 2.62\ \text{s}$$

As the periods are measured to a resolution of 0.01 s, all times are to the second decimal; it is proper to put this mean period also to the second decimal.

The errors in the measurements are

2.63 s – 2.62 s = 0.01 s
2.56 s – 2.62 s = – 0.06 s
2.42 s – 2.62 s = – 0.20 s
2.71 s – 2.62 s = 0.09 s
2.80 s – 2.62 s = 0.18 s

Note that the errors have the same units as the quantity to be measured.

The arithmetic mean of all the absolute errors (for arithmetic mean, we take only the magnitudes) is

$$\Delta T_{mean} = [(0.01+0.06+0.20+0.09+0.18)\text{s}]/5$$
$$= 0.54\ \text{s}/5$$
$$= 0.11\ \text{s}$$

That means, the period of oscillation of the simple pendulum is (2.62 ± 0.11) s i.e. it lies between (2.62 + 0.11) s and (2.62 – 0.11) s or between 2.73 s and 2.51 s. As the arithmetic mean of all the absolute errors is 0.11 s, there is already an error in the tenth of a second. Hence there is no point in giving the period to a hundredth. A more correct way will be to write

$$T = 2.6 \pm 0.1\ \text{s}$$

Note that the last numeral 6 is unreliable, since it may be anything between 5 and 7. We indicate this by saying that the measurement has two significant figures. In this case, the two significant figures are 2, which is reliable and 6, which has an error associated with it. You will learn more about the significant figures in section 2.7.

For this example, the relative error or the percentage error is

$$\delta a = \frac{0.1}{2.6}\times 100 = 4\%$$ ◀

2.6.2 Combination of Errors

If we do an experiment involving several measurements, we must know how the errors in all the measurements combine. For example,

How will you measure the length of a line?

What a naïve question, at this stage, you might say! But what if it is not a straight line? Draw a zigzag line in your copy, or on the blackboard. Well, not too difficult again. You might take a thread, place it along the line, open up the thread, and measure its length.

Now imagine that you want to measure the length of a national highway, a river, the railway track between two stations, or the boundary between two states or two nations. If you take a string of length 1 metre or 100 metre, keep it along the line, shift its position every time, the arithmetic of man-hours of labour and expenses on the project is not commensurate with the outcome. Moreover, errors are bound to occur in this enormous task. There is an interesting fact about this. France and Belgium share a common international boundary, whose length mentioned in the official documents of the two countries differs substantially!

Go one step beyond and imagine the coastline where land meets sea. Roads and rivers have fairly mild bends as compared to a coastline. Even so, all documents, including our school books, contain information on the length of the coastline of Gujarat or Andhra Pradesh, or the common boundary between two states, etc. Railway tickets come with the distance between stations printed on them. We have 'milestones' all along the roads indicating the distances to various towns. So, how is it done?

One has to decide how much error one can tolerate and optimise cost-effectiveness. If you want smaller errors, it will involve high technology and high costs. Suffice it to say that it requires fairly advanced level of physics, mathematics, engineering and technology. It belongs to the areas of fractals, which has lately become popular in theoretical physics. Even then one doesn't know how much to rely on the figure that props up, as is clear from the story of France and Belgium. Incidentally, this story of the France-Belgium discrepancy appears on the first page of an advanced Physics book on the subject of fractals and chaos!

mass density is obtained by deviding mass by the volume of the substance. If we have errors in the measurement of mass and of the sizes or dimensions, we must know what the error will be in the density of the substance. To make such estimates, we should learn how errors combine in various mathematical operations. For this, we use the following procedure.

(a) Error of a sum or a difference

Suppose two physical quantities A and B have measured values $A \pm \Delta A$, $B \pm \Delta B$ respectively where ΔA and ΔB are their absolute errors. We wish to find the error ΔZ in the sum

$$Z = A + B.$$

We have by addition, $Z \pm \Delta Z$

$$= (A \pm \Delta A) + (B \pm \Delta B).$$

The maximum possible error in Z

$$\Delta Z = \Delta A + \Delta B$$

For the difference $Z = A - B$, we have

$$Z \pm \Delta Z = (A \pm \Delta A) - (B \pm \Delta B)$$
$$= (A - B) \pm \Delta A \pm \Delta B$$

or, $\pm \Delta Z = \pm \Delta A \pm \Delta B$

The maximum value of the error ΔZ is again $\Delta A + \Delta B$.

Hence the rule : When two quantities are added or subtracted, the absolute error in the final result is the sum of the absolute errors in the individual quantities.

Example 2.8 The temperatures of two bodies measured by a thermometer are $t_1 = 20\ ^0C \pm 0.5\ ^0C$ and $t_2 = 50\ ^0C \pm 0.5\ ^0C$. Calculate the temperature difference and the error theirin.

Answer $t' = t_2 - t_1 = (50\ ^0C \pm 0.5\ ^0C) - (20^0C \pm 0.5\ ^0C)$

$t' = 30\ ^0C \pm 1\ ^0C$

(b) Error of a product or a quotient

Suppose $Z = AB$ and the measured values of A and B are $A \pm \Delta A$ and $B \pm \Delta B$. Then

$$Z \pm \Delta Z = (A \pm \Delta A)\ (B \pm \Delta B)$$
$$= AB \pm B\ \Delta A \pm A\ \Delta B \pm \Delta A\ \Delta B.$$

Dividing LHS by Z and RHS by AB we have,

$1 \pm (\Delta Z/Z) = 1 \pm (\Delta A/A) \pm (\Delta B/B) \pm (\Delta A/A)(\Delta B/B)$.

Since ΔA and ΔB are small, we shall ignore their product.

Hence the maximum relative error

$\Delta Z/\ Z = (\Delta A/A) + (\Delta B/B)$.

You can easily verify that this is true for division also.

Hence the rule : When two quantities are multiplied or divided, the relative error in the result is the sum of the relative errors in the multipliers.

Example 2.9 The resistance $R = V/I$ where $V = (100 \pm 5)$V and $I = (10 \pm 0.2)$A. Find the percentage error in R.

Answer The percentage error in V is 5% and in I it is 2%. The total error in R would therefore be 5% + 2% = 7%.

Example 2.10 Two resistors of resistances $R_1 = 100 \pm 3$ ohm and $R_2 = 200 \pm 4$ ohm are connected (a) in series, (b) in parallel. Find the equivalent resistance of the (a) series combination, (b) parallel combination. Use for (a) the relation $R = R_1 + R_2$, and for (b) $\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}$ and $\frac{\Delta R'}{R'^2} = \frac{\Delta R_1}{R_1^2} + \frac{\Delta R_2}{R_2^2}$

Answer (a) The equivalent resistance of series combination

$R = R_1 + R_2 = (100 \pm 3)$ ohm + (200 ± 4) ohm

$= 300 \pm 7$ ohm.

(b) The equivalent resistance of parallel combination

$$R' = \frac{R_1 R_2}{R_1 + R_2} = \frac{200}{3} = 66.7 \text{ ohm}$$

Then, from $\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}$

we get,

$$\frac{\Delta R'}{R'^2} = \frac{\Delta R_1}{R_1^2} + \frac{\Delta R_2}{R_2^2}$$

$$\Delta R' = \left(R'^2\right)\frac{\Delta R_1}{R_1^2} + \left(R'^2\right)\frac{\Delta R_2}{R_2^2}$$

$$= \left(\frac{66.7}{100}\right)^2 3 + \left(\frac{66.7}{200}\right)^2 4$$

$= 1.8$

Then, $R' = 66.7 \pm 1.8$ ohm

(Here, ΔR is expresed as 1.8 instead of 2 to keep in confirmity with the rules of significant figures.)

(c) Error in case of a measured quantity raised to a power

Suppose $Z = A^2$,

Then,

$$\Delta Z/Z = (\Delta A/A) + (\Delta A/A) = 2\,(\Delta A/A).$$

Hence, the relative error in A^2 is two times the error in A.

In general, if $Z = A^p B^q / C^r$

Then,

$$\Delta Z/Z = p\,(\Delta A/A) + q\,(\Delta B/B) + r\,(\Delta C/C).$$

Hence the rule : The relative error in a physical quantity raised to the power k is the k times the relative error in the individual quantity.

Example 2.11 Find the relative error in Z, if $Z = A^4 B^{1/3} / C D^{3/2}$.

Answer The relative error in Z is $\Delta Z/Z = 4(\Delta A/A) + (1/3)\,(\Delta B/B) + (\Delta C/C) + (3/2)\,(\Delta D/D)$.

Example 2.12 The period of oscillation of a simple pendulum is $T = 2\pi\sqrt{L/g}$. Measured value of L is 20.0 cm known to 1 mm accuracy and time for 100 oscillations of the pendulum is found to be 90 s using a wrist watch of 1 s resolution. What is the accuracy in the determination of g ?

Answer $g = 4\pi^2 L/T^2$

Here, $T = \frac{t}{n}$ and $\Delta T = \frac{\Delta t}{n}$. Therefore, $\frac{\Delta T}{T} = \frac{\Delta t}{t}$.

The errors in both L and t are the least count errors. Therefore,

$(\Delta g/g) = (\Delta L/L) + 2(\Delta T/T)$

$$= \frac{0.1}{20.0} + 2\left(\frac{1}{90}\right) = 0.027$$

Thus, the percentage error in g is

$100\,(\Delta g/g) = 100(\Delta L/L) + 2 \times 100\,(\Delta T/T)$

$= 3\%$

2.7 SIGNIFICANT FIGURES

As discussed above, every measurement involves errors. Thus, the result of measurement should be reported in a way that indicates the precision of measurement. Normally, the reported result of measurement is a number that includes all digits in the number that are known reliably plus the first digit that is uncertain. The reliable digits plus the first uncertain digit are known as **significant digits** or **significant figures**. If we say the period of oscillation of a simple pendulum is 1.62 s, the digits 1 and 6 are reliable and certain, while the digit 2 is uncertain. Thus, the measured value has three significant figures. The length of an object reported after measurement to be 287.5 cm has four significant figures, the digits 2, 8, 7 are certain while the digit 5 is uncertain. Clearly, reporting the result of measurement that includes more digits than the significant digits is superfluous and also misleading since it would give a wrong idea about the precision of measurement.

The rules for determining the number of significant figures can be understood from the following examples. Significant figures indicate, as already mentioned, the precision of measurement which depends on the least count of the measuring instrument. **A choice of change of different units does not change the number of significant digits or figures in a measurement.** This important remark makes most of the following observations clear:

(1) For example, the length 2.308 cm has four significant figures. But in different units, the same value can be written as 0.02308 m or 23.08 mm or 23080 μm.

All these numbers have the same number of significant figures (digits 2, 3, 0, 8), namely four. This shows that the location of decimal point is of no consequence in determining the number of significant figures.

The example gives the following rules :

- **All the non-zero digits are significant.**
- **All the zeros between two non-zero digits are significant, no matter where the decimal point is, if at all.**
- **If the number is less than 1, the zero(s) on the right of decimal point but to the left of the first non-zero digit are not significant.** [In $\underline{0.00}$ 2308, the underlined zeroes are not significant].
- **The terminal or trailing zero(s) in a number without a decimal point are not significant.**

  [Thus 123 m = 12300 cm = 123000 mm has *three* significant figures, the trailing zero(s) being not significant.] However, you can also see the next observation.
- **The trailing zero(s) in a number with a decimal point are significant.**

  [The numbers 3.500 or 0.06900 have four significant figures each.]

(2) There can be some confusion regarding the trailing zero(s). Suppose a length is reported to be 4.700 m. It is evident that the zeroes here are meant to convey the precision of measurement and are, therefore, significant. [If these were not, it would be superfluous to write them explicitly, the reported measurement would have been simply 4.7 m]. Now suppose we change units, then

4.700 m = 470.0 cm = 4700 mm = 0.004700 km

Since the last number has trailing zero(s) in a number with no decimal, we would conclude erroneously from observation (1) above that the number has *two* significant figures, while in fact, it has four significant figures and a mere change of units cannot change the number of significant figures.

(3) **To remove such ambiguities in determining the number of significant figures, the best way is to report every measurement in scientific notation (in the power of 10).** In this notation, every number is expressed as $a \times 10^b$, where a is a number between 1 and 10, and b is any positive or negative exponent (or power) of 10. In order to get an approximate idea of the number, we may round off the number a to 1 (for $a \le 5$) and to 10 (for $5 < a \le 10$). Then the number can be expressed approximately as 10^b in which the exponent (or power) b of 10 is called **order of magnitude** of the physical quantity. When only an estimate is required, the quantity is of the order of 10^b. For example, the diameter of the earth (1.28×10^7m) is of the order of 10^7m with the order of magnitude 7. The diameter of hydrogen atom (1.06×10^{-10}m) is of the order of 10^{-10}m, with the order of magnitude –10. Thus, the diameter of the earth is 17 orders of magnitude larger than the hydrogen atom.

It is often customary to write the decimal after the first digit. Now the confusion mentioned in (a) above disappears :

$$4.700 \text{ m} = 4.700 \times 10^2 \text{ cm} = 4.700 \times 10^3 \text{ mm} = 4.700 \times 10^{-3} \text{ km}$$

The power of 10 is irrelevant to the determination of significant figures. However, all zeroes appearing in the base number in the scientific notation are significant. Each number in this case has *four* significant figures.

Thus, in the scientific notation, no confusion arises about the trailing zero(s) in the base number *a*. They are always significant.

(4) The scientific notation is ideal for reporting measurement. But if this is not adopted, we use the rules adopted in the preceding example :

- **For a number greater than 1, without any decimal, the trailing zero(s) are not significant.**
- **For a number with a decimal, the trailing zero(s) are significant.**

(5) The digit 0 conventionally put on the left of a decimal for a number less than 1 (like 0.1250) is never significant. However, the zeroes at the end of such number are significant in a measurement.

(6) The multiplying or dividing factors which are neither rounded numbers nor numbers representing measured values are exact and have infinite number of significant digits. For example in $r = \frac{d}{2}$ or $s = 2\pi r$, the factor 2 is an exact number and it can be written as 2.0, 2.00 or 2.0000 as required. Similarly, in $T = \frac{t}{n}$, *n* is an exact number.

2.7.1 Rules for Arithmetic Operations with Significant Figures

The result of a calculation involving approximate measured values of quantities (i.e. values with limited number of significant figures) must reflect the uncertainties in the original measured values. It cannot be more accurate than the original measured values themselves on which the result is based. In general, the final result should not have more significant figures than the original data from which it was obtained. Thus, if mass of an object is measured to be, say, 4.237 g (four significant figures) and its volume is measured to be 2.51 cm^3, then its density, by mere arithmetic division, is 1.68804780876 g/cm^3 upto 11 decimal places. It would be clearly absurd and irrelevant to record the calculated value of density to such a precision when the measurements on which the value is based, have much less precision. The following rules for arithmetic operations with significant figures ensure that the final result of a calculation is shown with the precision that is consistent with the precision of the input measured values :

(1) **In multiplication or division, the final result should retain as many significant figures as are there in the original number with the least significant figures.**

Thus, in the example above, density should be reported to *three* significant figures.

$$\text{Density} = \frac{4.237\,\text{g}}{2.51\ \text{cm}^3} = 1.69\ \text{g cm}^{-3}$$

Similarly, if the speed of light is given as 3×10^8 m s^{-1} (one significant figure) and one year (1y = 365.25 d) has 3.1557×10^7 s (*five* significant figures), the light year is 9.47×10^{15} m (*three* significant figures).

(2) **In addition or subtraction, the final result should retain as many decimal places as are there in the number with the least decimal places.**

For example, the sum of the numbers 436.32 g, 227.2 g and 0.301 g by mere arithmetic addition, is 663.821 g. But the least precise measurement (227.2 g) is correct to only one decimal place. The final result should, therefore, be rounded off to 663.8 g.

Similarly, the difference in length can be expressed as :

0.307 m – 0.304 m = 0.003 m = 3×10^{-3} m.

Note that we should not use the *rule* (1) applicable for multiplication and division and write 664 g as the result in the example of **addition** and 3.00×10^{-3} m in the example of **subtraction**. They do not convey the precision of measurement properly. For addition and subtraction, the rule is in terms of decimal places.

2.7.2 Rounding off the Uncertain Digits

The result of computation with approximate numbers, which contain more than one uncertain digit, should be rounded off. The rules for rounding off numbers to the appropriate significant figures are obvious in most cases. A number 2.74<u>6</u> rounded off to three significant figures is 2.75, while the number 2.743 would be 2.74. The *rule* by convention is that the **preceding digit is raised by 1 if the**

insignificant digit to be dropped (the underlined digit in this case) is more than 5, and is left unchanged if the latter is less than 5. But what if the number is 2.745 in which the insignificant digit is 5. Here, the convention is that **if the preceding digit is even, the insignificant digit is simply dropped and, if it is odd, the preceding digit is raised by** 1. Then, the number 2.745 rounded off to three significant figures becomes 2.74. On the other hand, the number 2.735 rounded off to three significant figures becomes 2.74 since the preceding digit is odd.

In any involved or complex multi-step calculation, you should retain, in intermediate steps, one digit more than the significant digits and round off to proper significant figures at the end of the calculation. Similarly, a number known to be within many significant figures, such as in 2.99792458×10^8 m/s for the speed of light in vacuum, is rounded off to an approximate value 3×10^8 m/s, which is often employed in computations. Finally, remember that exact numbers that appear in formulae like 2π in $T = 2\pi\sqrt{\frac{L}{g}}$, have a large (infinite) number of significant figures. The value of π = 3.1415926.... is known to a large number of significant figures. You may take the value as 3.142 or 3.14 for π, with limited number of significant figures as required in specific cases.

Example 2.13 Each side of a cube is measured to be 7.203 m. What are the total surface area and the volume of the cube to appropriate significant figures?

Answer The number of significant figures in the measured length is 4. The calculated area and the volume should therefore be rounded off to 4 significant figures.

Surface area of the cube $= 6(7.203)^2\ m^2$
$= 311.299254\ m^2$
$= 311.3\ m^2$

Volume of the cube $= (7.203)^3\ m^3$
$= 373.714754\ m^3$
$= 373.7\ m^3$

Example 2.14 5.74 g of a substance occupies 1.2 cm^3. Express its density by keeping the significant figures in view.

Answer There are 3 significant figures in the measured mass whereas there are only 2 significant figures in the measured volume. Hence the density should be expressed to only 2 significant figures.

$$\text{Density} = \frac{5.74}{1.2}\ \text{g cm}^{-3}$$
$$= 4.8\ \text{g cm}^{-3}.$$

2.7.3 Rules for Determining the Uncertainty in the Results of Arithmatic Calculations

The rules for determining the uncertainty or error in the number/measured quantity in arithmetic operations can be understood from the following examples.

(1) If the length and breadth of a thin rectangular sheet are measured, using a metre scale as 16.2 cm and, 10.1 cm respectively, there are three significant figures in each measurement. It means that the length l may be written as

$$l = 16.2 \pm 0.1\ \text{cm}$$
$$= 16.2\ \text{cm} \pm 0.6\ \%.$$

Similarly, the breadth b may be written as

$$b = 10.1 \pm 0.1\ \text{cm}$$
$$= 10.1\ \text{cm} \pm 1\ \%$$

Then, the error of the product of two (or more) experimental values, using the combination of errors rule, will be

$$l\,b = 163.62\ \text{cm}^2 \pm 1.6\%$$
$$= 163.62 \pm 2.6\ \text{cm}^2$$

This leads us to quote the final result as

$$l\,b = 164 \pm 3\ \text{cm}^2$$

Here 3 cm^2 is the uncertainty or error in the estimation of area of rectangular sheet.

(2) **If a set of experimental data is specified to n significant figures, a result obtained by combining the data will also be valid to n significant figures.**

However, if data are subtracted, the number of significant figures can be reduced.

For example, 12.9 g – 7.06 g, both specified to three significant figures, cannot properly be evaluated as 5.84 g but only as 5.8 g, as uncertainties in subtraction or addition combine in a different fashion (smallest number of decimal places rather than the number of significant figures in any of the number added or subtracted).

(3) **The relative error of a value of number specified to significant figures depends not only on n but also on the number itself.**

For example, the accuracy in measurement of mass 1.02 g is ± 0.01 g whereas another measurement 9.89 g is also accurate to ± 0.01 g. The relative error in 1.02 g is

$= (\pm 0.01/1.02) \times 100\ \%$

$= \pm 1\%$

Similarly, the relative error in 9.89 g is

$= (\pm 0.01/9.89) \times 100\ \%$

$= \pm 0.1\ \%$

Finally, remember that **intermediate results in a multi-step computation should be calculated to one more significant figure in every measurement than the number of digits in the least precise measurement.** These should be justified by the data and then the arithmetic operations may be carried out; otherwise rounding errors can build up. For example, the reciprocal of 9.58, calculated (after rounding off) to the same number of significant figures (three) is 0.104, but the reciprocal of 0.104 calculated to three significant figures is 9.62. However, if we had written 1/9.58 = 0.1044 and then taken the reciprocal to three significant figures, we would have retrieved the original value of 9.58.

This example justifies the idea to retain one more extra digit (than the number of digits in the least precise measurement) in intermediate steps of the complex multi-step calculations in order to avoid additional errors in the process of rounding off the numbers.

2.8 DIMENSIONS OF PHYSICAL QUANTITIES

The nature of a physical quantity is described by its dimensions. All the physical quantities represented by derived units can be expressed in terms of some combination of seven fundamental or base quantities. We shall call these base quantities as the seven dimensions of the physical world, which are denoted with square brackets [ ]. Thus, length has the dimension [L], mass [M], time [T], electric current [A], thermodynamic temperature [K], luminous intensity [cd], and amount of substance [mol]. **The dimensions of a physical quantity are the powers (or exponents) to which the base quantities are raised to represent that quantity**. Note that using the square brackets [ ] round a quantity means that we are dealing with '**the dimensions of**' the quantity.

In mechanics, all the physical quantities can be written in terms of the dimensions [L], [M] and [T]. For example, the volume occupied by an object is expressed as the product of length, breadth and height, or three lengths. Hence the dimensions of volume are $[L] \times [L] \times [L] = [L]^3 = [L^3]$. As the volume is independent of mass and time, it is said to possess zero dimension in mass [M°], zero dimension in time [T°] and three dimensions in length.

Similarly, force, as the product of mass and acceleration, can be expressed as

Force = mass × acceleration

= mass × (length)/(time)^2

The dimensions of force are $[M]\ [L]/[T]^2 = [M\ L\ T^{-2}]$. Thus, the force has one dimension in mass, one dimension in length, and –2 dimensions in time. The dimensions in all other base quantities are zero.

Note that in this type of representation, the magnitudes are not considered. It is the quality of the type of the physical quantity that enters. Thus, a change in velocity, initial velocity, average velocity, final velocity, and speed are all equivalent in this context. Since all these quantities can be expressed as length/time, their dimensions are [L]/[T] or $[L\ T^{-1}]$.

2.9 DIMENSIONAL FORMULAE AND DIMENSIONAL EQUATIONS

The expression which shows how and which of the base quantities represent the dimensions of a physical quantity is called the *dimensional formula* of the given physical quantity. For example, the dimensional formula of the volume is $[M^\circ\ L^3\ T^\circ]$, and that of speed or velocity is $[M^\circ\ L\ T^{-1}]$. Similarly, $[M^\circ\ L\ T^{-2}]$ is the dimensional formula of acceleration and $[M\ L^{-3}\ T^\circ]$ that of mass density.

An equation obtained by equating a physical quantity with its dimensional formula is called the **dimensional equation** of the physical