PROBLEM

Discuss the monotonocity of $Q(x)$, where

$$Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)\ \forall\ x \in R$$

It is given that $f''(x) > 0\ \forall\ x \in R$. Find also the points of maxima and minima of $Q(x)$.

SOLUTION

Given $Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$$\therefore Q'(x) = 2xf'\left(\frac{x^2}{2}\right) - 2xf'(6 - x^2)$$

$$= 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right\}$$

But given that $f''(x) > 0$. Thus, $f'(x)$ is increasing for all $x \in R$.

Case I: Let $\frac{x^2}{2} > (6 - x^2)$ or $x^2 > 4$

$\therefore x \in (-\infty, -2) \cup (2, \infty)$

or $f\left(\frac{x^2}{2}\right) > f(6 - x^2)$

or $f\left(\frac{x^2}{2}\right) - f(6 - x^2) > 0$

If $x > 0$, then $Q'(x) > 0$ or $x \in (2, \infty)$.

If $x > 0$, then $Q'(x) < 0$ or $x \in (-\infty, -2)$.

Case II: Let $\frac{x^2}{2} < (6 - x^2)$ or $x^2 < 4$ or $x \in (-2, 2)$

or $f' = \left(\frac{x^2}{2}\right) < f'(6 - x^2)$ or $f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) < 0$

If $x > 0$, then $Q'(x) < 0$ or $x \in (0, 2)$.
If $x < 0$, then $Q'(x) > 0$ or $x \in (-2, 0)$.
Combining both cases, $Q(x)$ is increasing in $x \in (-2, 0) \cup (2, \infty)$, and $Q(x)$ is decreasing in $x \in (-\infty, -2) \cup (0, 2)$.