PROBLEM

Let $f:\ (0,\infty) \to R$ be given by $f(x)\int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$. Then

(JEE Adv. 2014)

(a) $f(x)$ is monotonically increasing on $[1,\infty)$

(b) $f(x)$ is monotonically decreasing on $(0, 1)$

(c) $f(x)+f\left(\frac{1}{x}\right)=0$, for all $x\in(0,\infty)$

(d) $f(2^x)$ is an odd function of x on R

SOLUTION

(a, c, d) $f(x)=\int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$

$$\therefore\ f'(x)=\frac{e^{-\left(x+\frac{1}{x}\right)}}{x}+\frac{x}{x^2}\ e^{-\left(\frac{1}{x}+x\right)}=\frac{2}{x}\ e^{-\left(x+\frac{1}{x}\right)}$$

For $x\in[1,\infty)$, $f'(x)>0$

$\therefore f$ is monotonically increasing on $[1,\infty)$

(a) is correct.

For $x\in(0,1)$, $f'(x)>0$

$\therefore$ (b) is not correct

$$f(x)+f\left(\frac{1}{x}\right)=\int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}+\int_{x}^{1/x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}=0$$

$\therefore$ (c) is correct.

Replacing x by 2^x in $f(x)+f\left(\frac{1}{x}\right)=0$

We get $f(2^x)+f\left(2^{-x}\right)=0$ or $f(2^x)=-f(2^{-x})$

$\therefore\quad f(2^x)$ is an odd function.

$\therefore$ (d) is correct.