PROBLEM

19. If f, g, and h are differentiable functions of x and

$$\Delta(x) = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)' & (x^2 g)' & (x^2 h)' \end{vmatrix},$$

then prove that

$$\Delta'(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}.$$

SOLUTION

$(xf)' = xf' + f$

and $(x^2f)'' = (2xf + x^2f')' = 2 + 2xf' + x^2f''$

$$\therefore\ \Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ 2f + 4xf' + x^2 f'' & 2g + 4xg' + x^2 g'' & 2h + 4xh' + x^2 f'' \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and then $R_3 \to R_3 - 4R_2 - 2R_1$

$$\therefore\ \Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$

Taking x common from R_2 and multiplying with R_3, we get

$$\Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}.$$

$$\therefore\ \frac{d\Delta}{dx} = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}$$

$$+ \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$