PROBLEM

For a twice differentiable function $f(x)$, $g(x)$ is defined as $g(x) = (f'(x)^2 + f''(x))\ f(x)$ on $[a, e]$. If for $a < b < c < d < e, f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2,$ $f(e) = 0$ then find the minimum number of zeros of $g(x)$.

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SOLUTION

$$g(x) = (f'(x))^2 + f''(x) f(x) = \frac{d}{dx}(f(x) f'(x))$$

Let $h(x) = f(x) f'(x)$

Then, $f(x) = 0$ has four roots namely a, α, β, e

where $b < \alpha < c$ and $c < \beta < d$.

And $f'(x) = 0$ at three points k_1, k_2, k_3

where $a < k_1 < \alpha, \alpha < k_2 < \beta, \beta < k_3 < e$

[$\because$ Between any two roots of a polynomial function $f(x) = 0$ there lies atleast one root of $f'(x) = 0$]

$\therefore$ There are atleast 7 roots of $f(x) . f'(x) = 0$

$\Rightarrow$ There are atleast 6 roots of $\frac{d}{dx}(f(x) f'(x)) = 0$

i.e. of $g(x) = 0$